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Spaces of functions of mixed smoothness and approximation from hyperbolic crosses

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Abstract

We compare function spaces of dominating mixed smoothness and spaces of best approximation with respect to hyperbolic crosses.

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1. Introduction

The aim of the paper is to study the relations between classes of functions defined by rates of best approximation with respect to hyperbolic crosses and smoothness spaces either defined by Fourier analytical tools or defined by differences with a dominating mixed term. In a sense this article is a continuation of the paper by Lizorkin and Nikol'skij [17] taking into account new developments such as the interpolation characterization (of the approximation classes) and more refined

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spaces such as the two scales of spaces $S^r_{p,q}B(\mathbb{R}^2)$ and $S^r_{p,q}F(\mathbb{R}^2)$ of Besov–Lizorkin–Triebel type with dominating mixed smoothness. On the other hand, it serves as an orientation towards approximation by sampling from sparse grids which will be the subject of a forthcoming paper by the second named author.

Here we call the set

$$H_m = \{(\xi_1, \xi_2) : \exists r \in \{0, \dots, m\} \text{ s.t. } |\xi_1| \leqslant 2^r \pi \text{ and } |\xi_2| \leqslant 2^{m-r}\pi\} \tag{1}$$

the hyperbolic cross of order m, $m \in \mathbb{N}_0$. According to these sets we define the hyperbolic best approximation of order m in $L_p(\mathbb{R}^2)$ as

$$E_m(f, L_p) := \inf \|f - g|L_p(\mathbb{R}^2)\|, \tag{2}$$

where the infimum is taken with respect to all functions $g \in L_p(\mathbb{R}^2)$ such that the support of its Fourier transform $\mathscr{F}g$ is contained in H_m. This notion imitates those known from hyperbolic cross approximation of periodic functions or those from hyperbolic wavelet approximation, cf. [2,3,9,15,17,18,25,26]. For $r > 0$ and $1 \leqslant p,\ q \leqslant \infty$ the approximation space $A^r_{p,q}(\mathbb{R}^2)$ is defined as the collection of all $f \in L_p(\mathbb{R}^2)$ such that

$$|||f||| = \left(\sum_{m=0}^{\infty} 2^{mrq} E_m(f, L_p)^q \right)^{1/q} < \infty. \tag{3}$$

These classes have been introduced and investigated in [17]. However, in the literature the interest has been concentrated on the periodic case. A good source for those investigations and also concerning references is the book by Temlyakov [26]. We want to mention in this connection the paper of Burenkov and Gol'dman [7] where these authors developed a technique to transfer results from $\mathbb{R}^d$ to its periodic analog and vice versa under certain rather weak conditions. This could be used to get some of the following statements also in the periodic situation.

There are different ways to characterize approximation spaces. Usually, one tries to construct an appropriate modulus of smoothness. This has been done by DeVore et al. [11] in the situation considered here. Alternatively, following a general scheme due to DeVore and Popov [10], the approximation spaces defined in (3) can be characterized also as real interpolation spaces of couples of (fractional) Sobolev spaces with a dominating mixed derivative. This is more or less a folklore-type result but it paved the way to an application of interpolation arguments in the main part of our paper, cf. Section 5.

Our main interest is twofold. On the one hand, we are looking for optimal embeddings which relate the approximation spaces associated with (3) to the scales $S^r_{p,q}B(\mathbb{R}^2)$ and $S^r_{p,q}F(\mathbb{R}^2)$. Here we can improve earlier results of Lizorkin and Nikol'skij [17]. On the other hand, we deal with the approximation of functions with dominating mixed smoothness by partial sums $S^H_m f$ with respect to hyperbolic crosses. We are able to give a complete description of the asymptotic behavior of the rate of convergence to f in the L_p-norm if the function f belongs to one of these spaces $S^r_{p,q}B(\mathbb{R}^2)$ or $S^r_{p,q}F(\mathbb{R}^2)$. Here we extend results by Bugrov [7], Nikol'skaya

[18], Lizorkin and Nikol'skij [17], and Temlyakov [25,26]. In both cases it turns out that there is a rather sophisticated interplay with the microscopic parameter q. Surprisingly, the largest subspace within the above scales contained in $A^r_{p,\infty}(\mathbb{R}^2)$ is always a space of Lizorkin–Triebel type.

This paper concentrates on the bivariate situation. Here our intention was to reduce the technicalities and to increase the transparency of the arguments. Moreover, we have the convenient reference [21] at hand (which concentrates on $d = 2$, too). Let us emphasize that the methods will apply also to the general situation but we have not checked all details.

The paper is organized as follows. In Section 2 we recall the definition, properties and some equivalent characterizations of Besov–Lizorkin–Triebel classes of dominating mixed smoothness. In particular, we are concerned with embeddings. Section 3 is devoted to the description of approximation spaces with respect to the hyperbolic cross and its characterization as interpolation spaces. Next, in Section 4, we collect some results on the interpolation of spaces of dominating mixed smoothness which will turn out to be useful later on. The heart of the paper consists in a detailed comparison of these three scales given in Section 5. Finally, we investigate $\|f - S^H_m f | L_p(\mathbb{R}^2)\|$ for functions f belonging to either $S^r_{p,q}B(\mathbb{R}^2)$ or $S^r_{p,q}F(\mathbb{R}^2)$ whenever they are not contained in $A^r_{p,\infty}(\mathbb{R}^2)$.

2. Besov–Lizorkin–Triebel classes of dominating mixed smoothness

2.1. *Definition and some basic properties*

Here we follow [21, Chapter 2] and introduce the scales of Besov–Nikol'skij and Lizorkin–Triebel spaces of dominating mixed smoothness via the Fourier analytic approach.

Let $\mathbb{R}^n$ be the Euclidean n-space, by $\mathbb{N}$ we denote the natural numbers, $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$ and by $\mathbb{Z}$ the integers. We write $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1} a \leqslant b \leqslant ca.$$

As usual, $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ denote the Schwartz space of infinitely differentiable and rapidly decreasing functions and its dual, the space of tempered distributions, respectively. $\mathscr{F}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ its inverse, both extended to $\mathscr{S}'$. If necessary we indicate the dimension of the underlying Euclidean space like $\mathscr{F}_1$ denoting the Fourier transform on $\mathbb{R}$ in that way.

We shall use smooth dyadic decompositions of unity. Let φ_0 be an infinitely differentiable function such that $0 \leqslant \varphi_0(t) \leqslant 1$, $\varphi_0(t) = 1$ if $|t| \leqslant 1$, and $\varphi_0(t) = 0$ if $|t| > 3/2$. Then we put

$$\varphi(t) = \varphi_0(t/2) - \varphi_0(t), \quad \varphi_j(t) = \varphi(2^{-j+1} t), \quad j = 1, 2, 3, \ldots \,. \tag{4}$$

Obviously

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1, \quad t \in \mathbb{R}$$

and

$$f(t) = \sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j(\xi)\mathscr{F}f(\xi)](t)$$

if $f \in \mathscr{S}'(\mathbb{R})$ (convergence in $\mathscr{S}'(\mathbb{R})$). Similarly,

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varphi_j(x_1)\varphi_k(x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2$$

and

$$f(x_1, x_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \mathscr{F}^{-1}[\varphi_j(\xi_1)\varphi_k(\xi_2)\mathscr{F}f(\xi_1, \xi_2)](x_1, x_2)$$

if $f \in \mathscr{S}'(\mathbb{R}^2)$ (convergence in $\mathscr{S}'(\mathbb{R}^2)$). For later use observe

$$\varphi_j(t) = 1 \quad \text{if } 32^{j-2} \leqslant |t| \leqslant 2^j, \quad j \geqslant 1. \tag{5}$$

We shall use the abbreviations

$$f_j = \mathscr{F}^{-1}[\varphi_j \mathscr{F}f] \quad \text{and} \quad f_{j,k} = \mathscr{F}^{-1}[\varphi_j(\xi_1)\varphi_k(\xi_2)\mathscr{F}f(\xi_1, \xi_2)]. \tag{6}$$

First, we recall the definition of the Besov and Lizorkin–Triebel classes from the Fourier-analytical point of view in the one-dimensional isotropic setting.

Definition 1. Let $-\infty < r < \infty$ and $1 \leqslant q \leqslant \infty$.

(i) If $1 \leqslant p \leqslant \infty$, then we put

$$B^r_{p,q}(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : ||f|B^r_{p,q}(\mathbb{R})|| = \left(\sum_{j=0}^{\infty} 2^{jrq} ||f_j|L_p(\mathbb{R})||^q \right)^{1/q} < \infty \right\}$$

if $q < \infty$ and

$$B^r_{p,\infty}(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : ||f|B^r_{p,\infty}(\mathbb{R})|| = \sup_{j=0,1,\ldots} 2^{jr} ||f_j|L_p(\mathbb{R})|| < \infty \right\}.$$

(ii) If $1 \leqslant p < \infty$, then we put

$$F^r_{p,q}(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : ||f|F^r_{p,q}(\mathbb{R})|| = \left\| \left(\sum_{j=0}^{\infty} 2^{jrq} |f_j|^q \right)^{1/q} \Big| L_p(\mathbb{R}) \right\| < \infty \right\},$$

if $q < \infty$ and

$$F^r_{p,\infty}(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : \|f|F^r_{p,\infty}(\mathbb{R})\| = \left\| \sup_{j=0,1,2,\ldots} 2^{jr}|f_j| \, |L_p(\mathbb{R}) \right\| < \infty \right\}.$$

Remark 1. We refer to the monographs of Peetre [20] and Triebel [27–29] for further information.

The classes of interest in this paper are the following.

Definition 2. Let $-\infty < r < \infty$ and $1 \leqslant q \leqslant \infty$.

(i) If $1 \leqslant p \leqslant \infty$, then we put

$$S^r_{p,q}B(\mathbb{R}^2) = \Bigg\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f|S^r_{p,q}B(\mathbb{R}^2)\|. = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{(j+k)rq} \|f_{j,k}|L_p(\mathbb{R}^2)\|^q \right)^{1/q} < \infty \Bigg\}$$

(usual modification if $q = \infty$).

(ii) If $1 \leqslant p < \infty$, then we put

$$S^r_{p,q}F(\mathbb{R}^2) = \Bigg\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f|S^r_{p,q}F(\mathbb{R}^2)\|. = \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{(j+k)rq} |f_{j,k}|^q \right)^{1/q} \Bigg| L_p(\mathbb{R}^2) \right\| < \infty \Bigg\}$$

(usual modification if $q = \infty$).

Remark 2. The spaces $S^r_{p,2}F(\mathbb{R}^2)$ are of peculiar interest. For $r \in \mathbb{N}$ and $1 < p < \infty$ an equivalent characterization is given by $f \in S^r_{p,2}F(\mathbb{R}^2)$ if and only if

$$f, D^{(r,0)}f, D^{(0,r)}f, D^{(r,r)}f \in L_p(\mathbb{R}^2).$$

If $r = 0$, then $f \in S^0_{p,2}F(\mathbb{R}^2)$ if and only if $f \in L_p(\mathbb{R}^2)$, cf. e.g. [21, Theorem 2.3.1]. Hence, $S^r_{p,2}F(\mathbb{R}^2)$ are Sobolev spaces with a dominating mixed derivative (in the sense of equivalent norms). If $r \in \mathbb{R}$ and $1 < p < \infty$, then

$$\|\mathscr{F}^{-1}(1 + |\xi_1|^2)^{r/2}(1 + |\xi_2|^2)^{r/2}\mathscr{F}f|L_p(\mathbb{R}^2)\|$$

represents an equivalent norm on $S^r_{p,2}F(\mathbb{R}^2)$. All spaces admit characterizations in terms of mixed derivatives and mixed differences if $r > 0$, cf. [21, 2.3.3, 2.3.4].

Remark 3. Both, $S^r_{p,q}B(\mathbb{R}^2)$ and $S^r_{p,q}F(\mathbb{R}^2)$ are Banach spaces. The norms are so-called cross norms. This means

$$\| f(x_1)g(x_2)|S^r_{p,q}B(\mathbb{R}^2)\| = \| f|B^r_{p,q}(\mathbb{R})\| \, \| g|B^r_{p,q}(\mathbb{R})\| \tag{7}$$

and

$$\| f(x_1)g(x_2)|S^r_{p,q}F(\mathbb{R}^2)\| = \| f|F^r_{p,q}(\mathbb{R})\| \, \| g|F^r_{p,q}(\mathbb{R})\|, \tag{8}$$

respectively. These relations are sometimes helpful for a better understanding.

Remark 4. As for the history of these spaces we refer also to Amanov [1], Nikol'skij [19], Lizorkin and Nikol'skij [17] and the survey Besov et al. [5].

Let us recall that some of these classes admit a so-called Lizorkin representation. In this situation the means $\mathscr{F}^{-1}[\varphi_j(\xi_1)\varphi_k(\xi_2)\mathscr{F}f(\xi_1,\xi_2)](\cdot)$ are replaced by $\mathscr{F}^{-1}[p_{j,k}(\xi_1,\xi_2)\mathscr{F}f(\xi_1,\xi_2)](\cdot)$, where the functions $p_{j,k}$ are defined as follows: let

$$\begin{aligned} P_{j,k} &= \{(x_1,x_2) : 2^{j-1}\pi < |x_1| \leqslant 2^j\pi,\ 2^{k-1}\pi < |x_2| \leqslant 2^k\pi\}, \quad j,k \in \mathbb{N}, \\ P_{j,0} &= \{(x_1,x_2) : 2^{j-1}\pi < |x_1| \leqslant 2^j\pi,\ |x_2| \leqslant \pi\}, \quad j \in \mathbb{N}, \\ P_{0,k} &= \{(x_1,x_2) : |x_1| \leqslant \pi,\ 2^{k-1}\pi < |x_2| \leqslant 2^k\pi\}, \quad k \in \mathbb{N}, \\ P_{0,0} &= [-\pi,\pi] \times [-\pi,\pi]. \end{aligned} \tag{9}$$

The corresponding characteristic functions are denoted by $p_{j,k}$. Obviously, the $P_{j,k}$ generate a pairwise disjoint covering of $\mathbb{R}^2$ and

$$H_m = \bigcup_{j+k\leqslant m} P_{j,k}. \tag{10}$$

Henceforth we shall use the abbreviation

$$\widetilde{f}_{j,k}(x) = \mathscr{F}^{-1}[p_{j,k}(\xi)\mathscr{F}f(\xi)](x). \tag{11}$$

Proposition 1. *Let* $-\infty < r < \infty$ *and* $1 < p < \infty$.

(i) *If* $1 \leqslant q \leqslant \infty$, *then*

$$S^r_{p,q}B(\mathbb{R}^2) = \Bigg\{ f \in \mathscr{S}'(\mathbb{R}^2) : \| f|S^r_{p,q}B(\mathbb{R}^2)\|. = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{(j+k)rq} \, \| \widetilde{f}_{j,k}|L_p(\mathbb{R}^2)\|^q \right)^{1/q} < \infty \Bigg\}$$

in the sense of equivalent norms.

(ii) *If* $1 < q < \infty$, *then*

$$S^r_{p,q}F(\mathbb{R}^2) = \Bigg\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f|S^r_{p,q}F(\mathbb{R}^2)\|. = \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{(j+k)rq} |\widetilde{f}_{j,k}|^q \right)^{1/q} \middle| L_p(\mathbb{R}^2) \right\| < \infty \Bigg\}$$

in the sense of equivalent norms.

Later on we shall make use of duality arguments. To have a concise formulation we introduce the closure of $\mathscr{S}(\mathbb{R}^2)$ in these scales. Let $s^r_{p,q}f(\mathbb{R}^2)$ be the closure of $\mathscr{S}(\mathbb{R}^2)$ in $S^r_{p,q}F(\mathbb{R}^2)$ and let $s^r_{p,q}b(\mathbb{R}^2)$ be the closure of $\mathscr{S}(\mathbb{R}^2)$ in $S^r_{p,q}B(\mathbb{R}^2)$, respectively. Of course, these new spaces are equipped with the induced norms. If $\max(p,q) < \infty$, then $s^r_{p,q}f(\mathbb{R}^2) = S^r_{p,q}F(\mathbb{R}^2)$ and $s^r_{p,q}b(\mathbb{R}^2) = S^r_{p,q}B(\mathbb{R}^2)$ holds, cf. [21, Theorem 2.2.4]. Without additional difficulties one can carry over the proofs of some duality assertions given in [28, 2.11] from isotropic Besov and Lizorkin–Triebel to the case considered here.

Proposition 2. *Suppose* $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, *and* $r \in \mathbb{R}$.

(i) *In the sense of the duality pairing between* $\mathscr{S}(\mathbb{R}^2)$ *and* $\mathscr{S}'(\mathbb{R}^2)$ *we can identify the dual space of* $s^r_{p,q}b(\mathbb{R}^2)$ *with* $S^{-r}_{p',q'}B(\mathbb{R}^2)$.

(ii) *Suppose* $1 < p < \infty$. *In the sense of the duality pairing between* $\mathscr{S}(\mathbb{R}^2)$ *and* $\mathscr{S}'(\mathbb{R}^2)$ *we can identify the dual space of* $s^r_{p,q}f(\mathbb{R}^2)$ *with* $S^{-r}_{p',q'}F(\mathbb{R}^2)$.

2.2. Horizontal embeddings

Here we are going to compare $S^r_{p_0,q_0}B(\mathbb{R}^2)$ and $S^r_{p_1,q_1}F(\mathbb{R}^2)$. In a $(r, 1/p)$-plane this corresponds to horizontal straight lines. Sobolev type embeddings would correspond to straightlines with slope 1 (in dimension one). For that reason we shall call them diagonal embeddings. They will be investigated in the next subsection.

Since the spaces are defined on the whole of $\mathbb{R}^2$ it is clear that these classes are incomparable for $p_0 \neq p_1$. For that reason we consider $p_0 = p_1$ only. Clearly, we have the monotonicity of these scales with respect to the microscopic parameter q. This will be used without further reference.

Theorem 1. *Let* $1 \leqslant p < \infty$, $1 \leqslant q$, $u \leqslant \infty$, *and* $r \in \mathbb{R}$.

(i) $S^r_{p,u}B(\mathbb{R}^2) \hookrightarrow S^r_{p,q}F(\mathbb{R}^2)$ *holds if and only if* $u \leqslant \min(p,q)$.

(ii) $S^r_{p,q}F(\mathbb{R}^2) \hookrightarrow S^r_{p,u}B(\mathbb{R}^2)$ *holds if and only if* $u \geqslant \max(p,q)$.

Proof. Sufficiency follows from the elementary inequalities

$$\| \cdot | \ell_{\max(p,q)}(L_p) \| \leqslant \| \cdot | L_p(\ell_q) \| \leqslant \| \cdot | \ell_{\min(p,q)}(L_p) \|. \tag{12}$$

Necessity of the given conditions can be reduced to the necessity of these conditions in the framework of the isotropic Besov and Lizorkin–Triebel spaces, see Remark 3 and [23]. □

2.3. Diagonal embeddings

Now we consider embeddings along lines with $r - 1/p = \text{const}$.

Theorem 2. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, *and* $r_0, r_1 \in \mathbb{R}$. *Suppose*

$$r_0 - \frac{1}{p_0} = r_1 - \frac{1}{p_1}.$$

(i) $S^{r_0}_{p_0,q_0}B(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,q_1}B(\mathbb{R}^2)$ *holds if and only if* $q_0 \leqslant q_1$.

(ii) *Let* $p_1 < \infty$. *Then* $S^{r_0}_{p_0,q_0}F(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,q_1}F(\mathbb{R}^2)$ *holds for any pair* q_0, q_1.

Proof. A proof of the sufficiency can be found in [21, 2.4.1]. The necessity in (i) becomes again a consequence of the sharpness of the corresponding assertion for Besov spaces $B^r_{p,q}(\mathbb{R})$, see [23], and Remark 3. □

It remains to consider the mixed problem.

Theorem 3. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, \ q_1 \leqslant \infty$, *and* $r_0, r_1 \in \mathbb{R}$. *Suppose*

$$r_0 - \frac{1}{p_0} = r_1 - \frac{1}{p_1}.$$

(i) *Let* $p_1 < \infty$. *Then* $S^{r_0}_{p_0,q_0}B(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,q_1}F(\mathbb{R}^2)$ *holds if and only if* $q_0 \leqslant p_1$.

(ii) *Suppose* $1 < p_0 < \infty$. *Then* $S^{r_0}_{p_0,q_0}F(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,q_1}B(\mathbb{R}^2)$ *holds if and only if* $p_0 \leqslant q_1$.

Proof. Necessity of these conditions are obtained from the corresponding assertions for Besov and Lizorkin–Triebel spaces, see [23], and Remark 3. Sufficiency in part (i) can be proved by employing Lemma 1 below. The proof of (ii) follows by a duality argument, cf. Proposition 2. □

Remark 5. As a consequence of the theorem it follows that under the above assumptions the embeddings

$$S^{r_0}_{p_0,p_1}B(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,1}F(\mathbb{R}^2)$$

and

$$S^{r_0}_{p_0,\infty}F(\mathbb{R}^2) \hookrightarrow S^{r_1}_{p_1,p_0}B(\mathbb{R}^2)$$

hold. These cannot be improved with respect to the microscopic parameters. In the isotropic context these embeddings have been proved by Jawerth [14] and Franke [13]. Both authors used real interpolation in a way which does not work here, cf. Section 4.1.

Here we formulate the nonperiodic counterpart of a result by Temlyakov [26, Lemma 2.2.1].

Lemma 1. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$. *Suppose* $f^{j,k} \in L_{p_0}(\mathbb{R}^2)$ *and*

$$\operatorname{supp} \mathscr{F} f^{j,k} \subset \{(\xi_1, \xi_2) \colon |\xi_1| \leqslant 2^j, \ |\xi_2| \leqslant 2^k\}, \quad j,k \in \mathbb{N}_0.$$

Then there exists a constant c such that

$$\left\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f^{j,k}(x)| \,\middle|\, L_{p_1}(\mathbb{R}^2) \right\| \leqslant c \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(2^{(j+k)(\frac{1}{p_0}-\frac{1}{p_1})} \| f^{j,k} | L_{p_0}(\mathbb{R}^2) \| \right)^{p_1} \right)^{1/p_1} \tag{13}$$

holds for all such sequences $\{f^{j,k}\}_{j,k}$.

Proof. Temlyakov works on the n-torus but this does not influence the argument. Only one additional remark has to be made. Temlyakov formulated a weaker result than he has proved. He stated (13) with

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f^{j,k}(x) \quad \text{instead of} \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f^{j,k}(x)|$$

on the left-hand side. However, his arguments apply also in the form given here, cf. in this connection [26, the formula at the bottom on p. 146]. □

Remark 6. Here we meet one of the situations where one could use the machinery of Burenkov and Gol'dman [7] to carry over a known estimate in the periodic case to its nonperiodic analog. However, the arguments used in the periodic situations have obvious nonperiodic counterparts.

3. Approximation with respect to the hyperbolic cross

3.1. Approximation by partial sums

Here we follow Lizorkin and Nikol'skij [17], but see also Temlyakov [25,26], DeVore et al. [9] and Kamont [15].

A basic role will be played by

$$S_m^H f(x) = \mathscr{F}^{-1}[\mathscr{X}_m(\xi)\mathscr{F}f(\xi)](x), \quad m \in \mathbb{N}_0, \ f \in L_p(\mathbb{R}^2), \tag{14}$$

where

$$\mathscr{X}_m(\xi) = \begin{cases} 1, & \xi \in H_m, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

In the language of Delvos and Schempp [8] the means $S_m^H f$ are the nonperiodic counterparts of the pseudo-hyperbolic dyadic Fourier approximation. Observe

$$S_m^H f = \sum_{j+k \leqslant m} \widetilde{f}_{j,k},$$

cf. (11). Important for us will be the following version of the Littlewood–Paley theorem: if $1<p<\infty$, then there exist positive constants A_p and B_p such that

$$A_p \|f|L_p(\mathbb{R}^2)\| \leqslant \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\widetilde{f}_{j,k}(x)|^2 \right)^{1/2} \middle| L_p(\mathbb{R}^2) \right\| \leqslant B_p \|f|L_p(\mathbb{R}^2)\| \tag{16}$$

cf. [16,19] and the comments in [17].

Next, we recall a characterization of the approximation classes, defined in (3), which was given in [17, Theorem 4.3]. We define a norm on $A^r_{p,q}$ by

$$\|f|A^r_{p,q}(\mathbb{R}^2)\| = \|f|L_p(\mathbb{R}^2)\| + |||f|||, \tag{17}$$

cf. (3). With this norm $A^r_{p,q}(\mathbb{R}^2)$ becomes a Banach space. However, often it is more convenient to work with $|||\cdot|||$ instead of the norm itself. If necessary, then we shall also write $|||\cdot|A^r_{p,q}(\mathbb{R}^2)|||$.

Proposition 3. *Let $1<p<\infty$, $1 \leqslant q \leqslant \infty$ and $r>0$. Then $A^r_{p,q}(\mathbb{R}^2)$ is the collection of all functions $f \in L_p(\mathbb{R}^2)$ such that*

$$\|f\|^{\#} = \left(\sum_{m=0}^{\infty} 2^{mrq} \|S^H_{m+1} f - S^H_m f|L_p(\mathbb{R}^2)\|^q \right)^{1/q} < \infty. \tag{18}$$

Moreover, there are constants A and B such that

$$A|||f||| \leqslant \|f\|^{\#} \leqslant B|||f||| \tag{19}$$

holds for all $f \in L_p(\mathbb{R}^2)$.

Remark 7. In fact, Lizorkin and Nikol'skij [17] proved a much more general assertion since they introduced more general approximation spaces (replacing the sequence of weights $\{2^{mr}\}_{m=0}^{\infty}$ by more general sequences).

Remark 8. Let us consider a specific class of functions in $A^r_{p,q}(\mathbb{R}^2)$. Suppose $f(x_1, x_2) = g(x_1)h(x_2)$ and supp $\mathscr{F}h \subset [-1, 1]$. Then $\widetilde{f}_{j,k} = 0$ if $k > 0$ and there exist positive constants c_1 and c_2 such that for all such functions f

$$c_1 ||g|B^r_{p,q}(\mathbb{R})|| \, ||h|L_p(\mathbb{R})|| \leqslant ||g(x_1)h(x_2)|A^r_{p,q}(\mathbb{R}^2)|| \leqslant c_2 ||g|B^r_{p,q}(\mathbb{R})|| \, ||h|L_p(\mathbb{R})|| \tag{20}$$

holds for $r > 0$. This indicates that $A^r_{p,q}(\mathbb{R}^2)$ will share several features with Besov spaces.

For $-\infty < \varrho < \infty$ we put

$$J_\varrho f = \mathscr{F}^{-1}[(1 + |\xi_1|^2)^{\varrho/2}(1 + |\xi_2|^2)^{\varrho/2}\mathscr{F}f(\xi_1, \xi_2)], \quad f \in \mathscr{S}'(\mathbb{R}^2).$$

Proposition 4 (Lift property). *Let $0 < r < \infty$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. If $r + \varrho > 0$, then $J_{-\varrho}$ is a linear and bounded one-to-one mapping from $A^r_{p,q}(\mathbb{R}^2)$ onto $A^{r+\varrho}_{p,q}(\mathbb{R}^2)$.*

Proof. We have the equivalence $(q < \infty)$

$$||J_{-\varrho} f|A^{r+\varrho}_{p,q}(\mathbb{R}^2)|| \sim \left(\sum_{m=0}^{\infty} 2^{rmq} \left|\left| \sum_{j+k=m} \mathscr{F}^{-1}[p_{j,k}\mathscr{F} 2^{\varrho(j+k)} J_{-\varrho} f] \right| L_p(\mathbb{R}^2) \right|\right|^q \right)^{1/q}, \tag{21}$$

cf. Proposition 3. Let φ_j, $j = 0, 1, \ldots$ be the system defined in (4) and put

$$\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad (\varphi_{-1} = 0).$$

It follows by (16) and the properties of the φ_j that

$$\left|\left| \left(\sum_{j+k=m} \mathscr{F}^{-1}[p_{j,k}\mathscr{F} 2^{\varrho(j+k)} J_{-\varrho} f] \right| L_p(\mathbb{R}^2) \right|\right|^q \Bigg)^{1/q}$$

$$\leqslant (A_p)^{-1} \left|\left| \left(\sum_{j+k=m} |\mathscr{F}^{-1}[2^{\varrho j}(1 + |\xi_1|^2)^{-\varrho/2} 2^{\varrho k}(1 + |\xi_2|^2)^{-\varrho/2} p_{j,k}\mathscr{F}f](x)|^2 \right)^{1/2} |L_p(\mathbb{R}^2) \right|\right|.$$

Using the identity

$$\begin{aligned} &\mathscr{F}^{-1}[2^{\varrho j}(1 + |\xi_1|^2)^{-\varrho/2} 2^{\varrho k}(1 + |\xi_2|^2)^{-\varrho/2} p_{j,k}\mathscr{F}f](x) \\ &\quad = \mathscr{F}^{-1}[2^{\varrho j}(1 + |\xi_1|^2)^{-\varrho/2}\widetilde{\varphi}_j(\xi_1) 2^{\varrho k}(1 + |\xi_2|^2)^{-\varrho/2}\widetilde{\varphi}_k(\xi_2)\mathscr{F}\widetilde{f}_{j,k}(\xi_1, \xi_2)](x) \end{aligned}$$

the right-hand side can be estimated by

$$c\left\|\left(\sum_{j+k=m}|\widetilde{f}_{j,k}|^2\right)^{1/2}\Big|L_p(\mathbb{R}^2)\right\|$$

using the vector-valued multiplier theorem (Theorem 1.10.3(ii)) in [21]. Inserting this into (21) we obtain

$$||J_{-\varrho}f|A^{r+\varrho}_{p,q}(\mathbb{R}^2)||\leqslant c||f|A^r_{p,q}(\mathbb{R}^2)||$$

using again (16). The same arguments give

$$||J_{\varrho}f|A^{r}_{p,q}(\mathbb{R}^2)||\leqslant c||f|A^{r+\varrho}_{p,q}(\mathbb{R}^2)||$$

and the proof is complete. □

3.2. *Real interpolation of approximation spaces*

Next, we shall make use of the interpolation theory of abstract approximation spaces, which is due to DeVore and Popov, cf. e.g. [10].

For basics of real interpolation we refer to Bergh and Löfström [4] and Triebel [27].

Proposition 5. *Let* $1<p<\infty$, $1\leqslant q\leqslant\infty$, *and* $r_0, r_1\geqslant 0$ *such that* $r_0\neq r_1$. *Let* $0<\Theta<1$. *We put* $r=(1-\Theta)r_0+\Theta r_1$. *Then*

$$(S^{r_0}_{p,2}F(\mathbb{R}^2), S^{r_1}_{p,2}F(\mathbb{R}^2))_{\Theta,q} = A^r_{p,q}(\mathbb{R}^2).$$

Proof. By the lift property of the spaces $A^r_{p,q}(\mathbb{R}^2)$ and $S^r_{p,2}F(\mathbb{R}^2)$ it is sufficient to consider the case $r_0=0$ and $r_1=r$. We define $X_0=\{0\}$ and

$$X_m=\{f\in L_p:\ \operatorname{supp}\mathscr{F}f\subset H_m\},\quad m=1,2,\ldots\ .$$

It is easily checked that this scale satisfies the assumptions in [10, Chapter 7, Section 5]. Based on the Lizorkin representation, cf. Proposition 1, and using the Littlewood–Paley assertion (16) we derive immediately a Jackson type inequality

$$\begin{aligned}||f-S^H_mf|L_p(\mathbb{R}^2)||&\leqslant (1/A_p)\left\|\left(\sum_{j+k>m}|\widetilde{f}_{j,k}|^2\right)^{1/2}\Big|L_p(\mathbb{R}^2)\right\|\\ &\leqslant c2^{-mr}||f|S^r_{p,2}F(\mathbb{R}^2)||\end{aligned}$$

valid for all $f\in S^r_{p,2}F(\mathbb{R}^2)$, $r>0$. Furthermore, by similar arguments the Bernstein type inequality

$$||f|S^r_{p,2}F(\mathbb{R}^2)||\leqslant c2^{mr}||f|L_p(\mathbb{R}^2)||$$

is valid for all $f \in X_m$. This is sufficient to apply the scheme of approximation spaces with $X = L_p(\mathbb{R}^2)$ and $Y = S^r_{p,2}F(\mathbb{R}^2)$, cf. [10, Chapter 7, Section 9]. $\square$

Remark 9. The counterparts in case of hyperbolic spline approximation on the cube and hyperbolic wavelet approximation on $\mathbb{R}^n$ have been proved in [9,10].

As a direct consequence of the preceding proposition one obtains the following result on interpolation of approximation spaces, cf. [10, Theorem 7.9.1, formula 7.9.7].

Proposition 6. *Let* $1<p<\infty$, $1\leqslant q,\ q_1\leqslant\infty$, *and* $0<r_1$. *Let* $0<\Theta<1$.

(i) *We put* $r=\Theta r_1$. *Then*
$$(L_p(\mathbb{R}^2), A^{r_1}_{p,q_1}(\mathbb{R}^2))_{\Theta,q} = A^r_{p,q}(\mathbb{R}^2).$$

(ii) *Let* $1\leqslant q_0\leqslant\infty$ *and* $0<r_0<r_1$. *We put* $r=(1-\Theta)r_0+\Theta r_1$. *Then*
$$(A^{r_0}_{p,q_0}(\mathbb{R}^2), A^{r_1}_{p,q_1}(\mathbb{R}^2))_{\Theta,q} = A^r_{p,q}(\mathbb{R}^2).$$

4. Complex and real interpolation of spaces with dominating mixed smoothness

Here we mention a few results about interpolation which will be useful in proving mixed diagonal embeddings.

4.1. Real interpolation

Proposition 7. *Let* $1\leqslant p\leqslant\infty$, $1\leqslant q_0,\ q_1\leqslant\infty$, *and* $r_0, r_1\in\mathbb{R}$. *Let* $0<\Theta<1$. *We put*

$$r=(1-\Theta)r_0+\Theta r_1 \quad \textit{and} \quad \frac{1}{q}=(1-\Theta)\frac{1}{q_0}+\Theta\frac{1}{q_1}.$$

Then

$$(S^{r_0}_{p,q_0}B(\mathbb{R}^2), S^{r_1}_{p,q_1}B(\mathbb{R}^2))_{\Theta,q} = S^r_{p,q}B(\mathbb{R}^2).$$

Proof. As in [27, 1.2.4,2.4.1,2.4.2] or [4, 6.4] the proof is reduced to the interpolation of certain vector-valued sequence spaces by means of retraction and coretraction.

Let $\{\varphi_j\}_j$ be the system of functions defined in (4). We put

$$\varrho_0(\xi) := \varphi_0(\xi) + \varphi_1(\xi)$$

$$\varrho_k(\xi) := \varphi_{k-1}(\xi) + \varphi_k(\xi) + \varphi_{k+1}(\xi), \quad k = 1, 2, \dots,$$

$$\varrho_{j,k}(\xi_1, \xi_2) = \varrho_j(\xi_1)\varrho_k(\xi_2), \quad j, k = 0, 1, \dots,$$

$$\varphi_{j,k}(\xi_1, \xi_2) = \varphi_j(\xi_1)\varphi_k(\xi_2), \quad j, k = 0, 1, \dots .$$

If $f \in \mathscr{S}'(\mathbb{R}^2)$ we define

$$Sf(x) := \left\{\mathscr{F}^{-1}[\varphi_{j,k}\mathscr{F}f](x)\right\}_{j,k=0}^{\infty}.$$

For $g_{j,k} \in \mathscr{S}'(\mathbb{R}^2)$, $j, k = 0, 1, \dots$ we define (formally)

$$R\{g_{j,k}\}_{j,k} := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \mathscr{F}^{-1}[\varrho_{j,k}\mathscr{F}g_{j,k}](x)$$

if the double series on the right-hand side converges in $\mathscr{S}'(\mathbb{R}^2)$. Obviously, we then have

$$R(Sf) = f.$$

We introduce the sequence spaces

$$\ell_q^r(\mathbb{N}_0, X) = \left\{ \{x_j\}_{j=0}^{\infty} \subset X\colon \ ||\{x_j\}_j|\ell_q^r(\mathbb{N}_0, X)|| = \left(\sum_{j=0}^{\infty} 2^{rjq}||x_j|X||^q\right)^{1/q} < \infty \right\}$$

and

$$\ell_q^r(\mathbb{N}_0^2, X) = \ell_q^r(\mathbb{N}_0, \ell_q^r(\mathbb{N}_0, X)),$$

where X is an arbitrary Banach space, $r \in \mathbb{R}$ and $1 \leqslant q \leqslant \infty$ (modification if $q = \infty$). Now, by definition of the spaces S is a linear and bounded operator

$$S : S_{p,q_i}^{r_i}B(\mathbb{R}^2) \to \ell_{q_i}^{r_i}(\mathbb{N}_0^2, L_p(\mathbb{R}^2)) \quad (i = 1, 2).$$

On the other hand, we have the boundedness of

$$R : \ell_{q_i}^{r_i}(\mathbb{N}_0^2, L_p(\mathbb{R}^2)) \to S_{p,q_i}^{r_i}B(\mathbb{R}^2) \quad (i = 1, 2).$$

This can be seen as follows. We observe that

$$||R\{g_{j,k}\}_{j,k}|S_{p,q_i}^{r_i}B(\mathbb{R}^2)||$$

$$\leqslant \sum_{|\ell|,|m| \leqslant 1} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{r_i(j+k)q_i} \left|\left|\mathscr{F}^{-1}[\varphi_{j,k}\varrho_{j+\ell,k+m}\mathscr{F}g_{j+\ell,k+m}]\right|L_p(\mathbb{R}^2)\right|\right|^{q_i} \right)^{1/q_i},$$

where $g_{j,k}$ and $\varrho_{j,k}$ are zero if $j<0$ or $k<0$. The L_p-norm on the right-hand side can be rewritten as the norm of a convolution and estimated by

$$c||g_{j+\ell,k+m}|L_p(\mathbb{R}^2)||$$

(where c does not depend on g, j, k) such that

$$\|R\{g_{j,k}\}_{j,k}|S^{r_i}_{p,q_i}B(\mathbb{R}^2)\| \leqslant c\|\{g_{j,k}\}_{j,k}|\ell^{r_i}_{q_i}(\mathbb{N}^2_0, L_p(\mathbb{R}^2))\|.$$

Let $r = (1-\Theta)r_0 + \Theta r_1$ and $1/q = (1-\Theta)/q_0 + \Theta/q_1$ and $X = L_p(\mathbb{R}^2)$. Let us recall the well-known formula ([4] or [27])

$$(\ell^{r_0}_{q_0}(\mathbb{N}_0, Y_0), \ell^{r_1}_{q_1}(\mathbb{N}_0, Y_1))_{\Theta,q} = \ell^r_q(\mathbb{N}_0, (Y_0, Y_1)_{\Theta,q}),$$

where (Y_0, Y_1) is an interpolation couple of Banach spaces. Iterated use of this formula leads to

$$(\ell^{r_0}_{q_0}(\mathbb{N}^2_0, X), \ell^{r_1}_{q_1}(\mathbb{N}^2_0, X))_{\Theta,q} = \ell^r_q(\mathbb{N}^2_0, X),$$

and proves the proposition. □

If one compares this result with the corresponding formula for Besov spaces

$$(B^{r_0}_{p,q_0}, B^{r_1}_{p,q_1})_{\Theta,q} = B^r_{p,q}, \quad r = (1-\Theta)r_0 + \Theta r_1, \quad r_0 \neq r_1,$$

cf. e.g. [4, Theorem 6.4.5], then the greater flexibility of this formula with respect to the microscopic parameters q_0, q_1 and q is obvious. For us it was a bit surprising that real interpolation connecting the classes of interest here requires much more restrictions. To support this we prove an instructive lemma. To have a concise formulation we introduce the following abbreviation. Let $S^r_{p,q}$ stand for one of the classes $S^r_{p,q}F(\mathbb{R}^2)$ and $S^r_{p,q}B(\mathbb{R}^2)$. If several spaces $S^{r_0}_{p_0,q_0}$, $S^{r_1}_{p_1,q_1}, \ldots$ occur, then they may be taken from different scales. Let A, B be an interpolation couple of Banach spaces and let $0 < \Theta < 1$. By (A, B, Θ) we denote an interpolation space of A and B which is generated by an interpolation functor of type Θ, cf. e.g. [27, 1.2.2]. In particular, this applies to real and complex interpolation with respect to the parameter Θ. Under these restrictions the following is true.

Lemma 2. *Suppose* $1 < p_0, p_1 < \infty$ *and* $r_0, r_1 \in \mathbb{R}$.

(i) *The embedding*

$$(S^{r_0}_{p_0,q_0}, S^{r_1}_{p_1,q_1}, \Theta) \hookrightarrow S^r_{p,q}$$

implies

$$\frac{1}{q} \leqslant \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

(ii) *If we replace in* (i) *one or two of the spaces* $S^{r_0}_{p_0,q_0}$, $S^{r_1}_{p_1,q_1}$ *or* $S^r_{p,q}$ *by the corresponding approximation spaces* $A^{r_0}_{p_0,q_0}(\mathbb{R}^2)$, $A^{r_1}_{p_1,q_1}(\mathbb{R}^2)$ *or* $A^r_{p,q}(\mathbb{R}^2)$, *respectively, then the conclusion is*

$$\frac{1}{\tilde{q}} \leqslant \frac{1-\Theta}{\tilde{q}_0} + \frac{\Theta}{\tilde{q}_1},$$

where

$$\tilde{q} := \begin{cases} q, & \textit{in case of no replacement}, \\ 2 & \textit{otherwise}. \end{cases}$$

Proof. We employ test functions defined as follows:

$$g_m(x_1, x_2) := \sum_{j=3}^{m-3} \alpha_j \mathscr{F}^{-1}[\psi(\xi_1 - c2^j, \xi_2 - c2^{m-j})](x), \quad m = 6, 7, \dots,$$

where $\psi \in \mathscr{S}(\mathbb{R}^2)$ and $\operatorname{supp} \mathscr{F}\psi \subset \{\xi : |\xi| \leqslant 1\}$. The constant $c = 7/8$ is chosen in such a way that it simplifies the norm of g_m as much as possible (cf. (5)). We have

$$||g_m|S^r_{p,q}F(\mathbb{R}^2)|| = ||g_m|S^r_{p,q}B(\mathbb{R}^2)|| = ||\mathscr{F}^{-1}\psi|L_p(\mathbb{R}^2)||2^{mr}\left(\sum_{j=3}^{m-3} |\alpha_j|^q\right)^{1/q}.$$

Further, there exist positive constants c_1 and c_2 such that

$$c_1|||g_m|A^r_{p,q}(\mathbb{R}^2)||| \leqslant 2^{rm}\left(\sum_{j=3}^{m-3} |\alpha_j|^2\right)^{1/2} \leqslant c_2|||g_m|A^r_{p,q}(\mathbb{R}^2)|||,$$

cf. Lemma 4. The interpolation property and the continuous embedding $(S^{r_0}_{p_0,q_0}, S^{r_1}_{p_1,q_1}, \Theta) \hookrightarrow S^r_{p,q}$ imply the inequality

$$\left(\sum_{j=3}^{m-3} |\alpha_j|^q\right)^{1/q} \leqslant c \left(\sum_{j=3}^{m-3} |\alpha_j|^{q_0}\right)^{\frac{1-\Theta}{q_0}} \left(\sum_{j=3}^{m-3} |\alpha_j|^{q_1}\right)^{\frac{\Theta}{q_1}}$$

with a constant c independent of m. This proves (i). The proof of (ii) is similar. □

There is a further interpolation formula well-known for the isotropic classes which has a counterpart in our situation.

Proposition 8. *Let* $1 < p_0 < p_1 < \infty$, $1 < q \leqslant \infty$, *and* $r \in \mathbb{R}$. *Let* $0 < \Theta < 1$. *We put* $1/p = (1-\Theta)(1/p_0) + \Theta(1/p_1)$. *Then*

$$(S^r_{p_0,q}F(\mathbb{R}^2), S^r_{p_1,q}F(\mathbb{R}^2))_{\Theta,p} = S^r_{p,q}F(\mathbb{R}^2). \tag{22}$$

Proof. We use the operators S and R defined in the proof of Proposition 7.

Let $1 < p < \infty$ and $1 < q \leqslant \infty$. It is obvious that the restriction of S is a linear bounded operator

$$S : S^0_{p,q}F(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2, \ell_q(\mathbb{N}^2_0)).$$

Moreover, it is clear by definition of the $\varrho_{j,k}$ that (at least formally)

$$R(Sf) = f \quad \text{for all } f \in S^0_{p,q}F(\mathbb{R}^2)$$

holds. We show that $R : L_p(\mathbb{R}^2, \ell_q) \mapsto S^0_{p,q}F(\mathbb{R}^2)$ is bounded. By using the properties of supp $\varphi_{j,k}$ we have

$$||R\{g_{j,k}\}_{j,k}|S^0_{p,q}F(\mathbb{R}^2)|| \leqslant \sum_{|\ell|,|m|\leqslant 1} \left|\left| \left(\sum_{j,k=0}^{\infty} |\mathscr{F}^{-1}[\varphi_{j,k}\varrho_{j+\ell,k+m}\mathscr{F} g_{j+\ell,k+m}](x)|^q \right)^{1/q} \right| L_p(\mathbb{R}^2) \right|\right|,$$

where $g_{j,k} \equiv 0$ if either $j<0$ or $k<0$. Applying a well-known assertion on convolutions, cf. [24, 2.2, p. 56/57] the right-hand side can be estimated from above by

$$c \left|\left| \left(\sum_{j,k=0}^{\infty} (Mg_{j,k})^q(x) \right)^{1/q} \right| L_p(\mathbb{R}^2) \right|\right|,$$

where M denotes the Hardy–Littlewood maximal function. The vector-valued maximal inequality of Fefferman and Stein [12] yields the desired boundedness if $1<q<\infty$. If $q = \infty$ we use

$$\left|\left| \sup_j \, (Mg_j)(x) \right| L_p(\mathbb{R}^2) \right|\right| \leqslant \left|\left| M\left(\sup_j \, |g_j| \right)(x) \right| L_p(\mathbb{R}^2) \right|\right|$$

and the scalar Hardy–Littlewood maximal inequality. Now, the desired formula in case $r = 0$ follows from

$$(L_{p_0}(A), L_{p_1}(A))_{\Theta,p} = L_p(A), \quad A = \ell_q(\mathbb{N}_0^2), \tag{23}$$

where $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ (cf. [27, Theorem 1.18.6/2] or [4, 5.1]). The general case $r\neq 0$ is a consequence of the lift property of the spaces under consideration, cf. [21, 2.2.6]. □

Remark 10. In addition to the proof of (22) let us note that

$$S : S^0_{p,1}F(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2, \ell_1), \quad 1 \leqslant p < \infty.$$

Moreover, (23) is true also for $A = \ell_1$. Hence, by interpolation

$$S : (S^0_{p_0,1}F(\mathbb{R}^2), S^0_{p_1,1}F(\mathbb{R}^2))_{\Theta,p} \mapsto L_p(\mathbb{R}^2, \ell_1).$$

This leads to

$$||f|S^0_{p,1}F(\mathbb{R}^2)|| = ||Sf|L_p(\mathbb{R}^2, \ell_1)|| \leqslant c||f|(S^0_{p_0,1}F(\mathbb{R}^2), S^0_{p_1,1}F(\mathbb{R}^2))_{\Theta,p}||.$$

In other words, using again the lift property of the spaces, we have the embedding

$$(S^r_{p_0,1}F(\mathbb{R}^2), S^r_{p_1,1}F(\mathbb{R}^2))_{\Theta,p} \hookrightarrow S^r_{p,1}F(\mathbb{R}^2), \quad \frac{1}{p} = (1-\Theta)\frac{1}{p_0} + \Theta\frac{1}{p_1}, \tag{24}$$

which will be sufficient for our later purposes as a replacement of (22) if $q = 1$.

4.2. Complex interpolation

For completeness we also state some results on complex interpolation.

Proposition 9. (i) *Let* $1 \leqslant p_0, p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, *and* $r_0, r_1 \in \mathbb{R}$. *Let* $0 < \Theta < 1$. *We put* $r = (1-\Theta)r_0 + \Theta r_1$,

$$\frac{1}{p} = (1-\Theta)\frac{1}{p_0} + \Theta\frac{1}{p_1} \quad \textit{and} \quad \frac{1}{q} = (1-\Theta)\frac{1}{q_0} + \Theta\frac{1}{q_1}.$$

Then

$$[S^{r_0}_{p_0,q_0}B(\mathbb{R}^2), S^{r_1}_{p_1,q_1}B(\mathbb{R}^2)]_\Theta = S^r_{p,q}B(\mathbb{R}^2).$$

(ii) *Let* $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 \leqslant \infty$, *and* $p, q, r_0, r_1, r, \Theta$ *as above. Then*

$$[S^{r_0}_{p,q_0}F(\mathbb{R}^2), S^{r_1}_{p,q_1}F(\mathbb{R}^2)]_\Theta = S^r_{p,q}F(\mathbb{R}^2).$$

Proof. As in proof of Proposition 7(i) is reduced to iterated use of the interpolation formula

$$[\ell^{r_0}_{q_0}(\mathbb{N}_0, X_0), \ell^{r_1}_{q_1}(\mathbb{N}_0, X_1)]_\Theta = \ell^r_q(\mathbb{N}_0, [X_0, X_1]_\Theta),$$

whereas (ii) is reduced to

$$[L_{p_0}(\mathbb{R}^2, Y_0), L_{p_1}(\mathbb{R}^2, Y_1)]_\Theta = L_p(\mathbb{R}^2, [Y_0, Y_1]_\Theta),$$

by means of the arguments in the proof of Proposition 8. □

5. A detailed comparison of $A^r_{p,q}(\mathbb{R}^2)$ with $S^r_{p,q}B(\mathbb{R}^2)$ and $S^r_{p,q}F(\mathbb{R}^2)$

This section is the heart of the paper. Here we clarify the somewhat complicated relations between the approximation classes on the one side and the spaces of Besov and Lizorkin–Triebel type on the other.

5.1. Preparations—classes of test functions

We shall investigate two different types of test functions. On the one hand, we study the nonperiodic counterpart of the Dirichlet kernel with respect to the hyperbolic annuli $H_{m+1} \backslash H_m$ and on the other hand the nonperiodic counterpart of lacunary series.

Recall, $p_{j,k}$ denotes the characteristic function of $P_{j,k}$.

Lemma 3. *Suppose* $1 < p < \infty$ *and* $1 \leqslant q \leqslant \infty$.

(i) *There exist positive constants* c_1 *and* c_2 *such that*

$$c_1 m^{1/p} 2^{m(1-1/p)} \leqslant \left\| \sum_{j=0}^{m} \mathscr{F}^{-1}[p_{j,m-j}] \middle| L_p(\mathbb{R}^2) \right\| \leqslant c_2 m^{1/p} 2^{m(1-1/p)} \tag{25}$$

holds for all $m \in \mathbb{N}$.

(ii) *There exist positive constants* c_1 *and* c_2 *such that*

$$c_1 m^{1/p} 2^{m(1-1/p)} \leqslant \left\| \sup_{0 \leqslant j \leqslant m} |\mathscr{F}^{-1}[p_{j,m-j}]| \middle| L_p(\mathbb{R}^2) \right\| \leqslant \left\| \sum_{j=0}^{m} |\mathscr{F}^{-1}[p_{j,m-j}]| \middle| L_p(\mathbb{R}^2) \right\| \leqslant c_2 m^{1/p} 2^{m(1-1/p)} \tag{26}$$

holds for all $m \in \mathbb{N}$.

(iii) *Define*

$$f_m(x) := \sum_{j=0}^{m} 2^{-rm} 2^{m(1/p-1)} \mathscr{F}^{-1} p_{j,m-j}, \quad m \in \mathbb{N}. \tag{27}$$

Then there exist positive constants c_1 *and* c_2 *such that*

$$c_1 m^{1/q} \leqslant \| f_m | S^r_{p,q} B(\mathbb{R}^2) \| \leqslant c_2 m^{1/q} \quad (r \in \mathbb{R}), \tag{28}$$

$$c_1 m^{1/p} \leqslant \| f_m | S^r_{p,q} F(\mathbb{R}^2) \| \leqslant c_2 m^{1/p} \quad (r \in \mathbb{R}) \tag{29}$$

and

$$c_1 m^{1/p} \leqslant \| | f_m | A^r_{p,q}(\mathbb{R}^2) \| | \leqslant c_2 m^{1/p} \quad (r > 0), \tag{30}$$

holds for all $m \in \mathbb{N}$.

Proof. Step 1: In what follows, we need the L_p-norm of the functions $\mathscr{F}^{-1} p_{j,k}$. Clearly, if $j, k \geqslant 1$ a homogeneity argument gives

$$\| \mathscr{F}^{-1} p_{j,k} | L_p(\mathbb{R}) \| = C 2^{j(1-1/p)} 2^{k(1-1/p)}. \tag{31}$$

A similar equality holds for the pairs $(0,k)$, $k \geqslant 1$ and $(j,0)$, $j \geqslant 1$. To prove the upper estimate we employ Lemma 1. For $1 \leqslant q < p < \infty$ it follows

$$\begin{aligned} \left\| \sum_{j=0}^{m} |\mathscr{F}^{-1} p_{j,m-j}| \middle| L_p(\mathbb{R}^2) \right\| &\leqslant c_1 \left(\sum_{j=0}^{m} 2^{m\left(\frac{1}{q}-\frac{1}{p}\right)p} \| \mathscr{F}^{-1} p_{j,m-j} | L_q(\mathbb{R}^2) \|^p \right)^{1/p} \\ &\leqslant c_2 \left(\sum_{j=0}^{m} 2^{m\left(\frac{1}{q}-\frac{1}{p}\right)p} 2^{m\left(1-\frac{1}{q}\right)p} \right)^{1/p} \\ &\leqslant c_3 m^{1/p} 2^{m\left(1-\frac{1}{p}\right)}. \end{aligned}$$

The estimate from below in the L_p-norm turns out to be a consequence of the estimate from above in case p', of the exactness of the L_2-result, an orthogonality argument and Hölder's inequality:

$$\begin{aligned}
(m+1)2^m\pi^2 &= \left\| \left| \sum_{j=0}^{m} \mathscr{F}^{-1}[p_{j,m-j}] \right| L_2(\mathbb{R}^2) \right\|^2 \\
&= \int \sum_{j=0}^{m} |\mathscr{F}^{-1}[p_{j,m-j}](x)|^2 \, dx \\
&\leqslant \int \sup_{0\leqslant j\leqslant m} |\mathscr{F}^{-1}[p_{j,m-j}](x)| \sum_{j=0}^{m} |\mathscr{F}^{-1}[p_{j,m-j}](x)| \, dx \\
&\leqslant \left\| \sup_{0\leqslant j\leqslant m} |\mathscr{F}^{-1}[p_{j,m-j}]| | L_p(\mathbb{R}^2) \right\| \left\| \sum_{j=0}^{m} |\mathscr{F}^{-1}[p_{j,m-j}]| | L_{p'}(\mathbb{R}^2) \right\| \\
&\leqslant c m^{1/p'} 2^{m(1-1/p')} \left\| \sup_{0\leqslant j\leqslant m} |\mathscr{F}^{-1}[p_{j,m-j}]| | L_p(\mathbb{R}^2) \right\|.
\end{aligned}$$

This proves (26) and at the same time (25).

Step 2: Due to Proposition 1 and (31) we have

$$\begin{aligned}
||f_m|S^r_{p,q}B(\mathbb{R}^2)|| &\sim \left(\sum_{j=0}^{m} 2^{m(1/p-1)q} ||\mathscr{F}^{-1}[p_{j,m-j}]|L_p(\mathbb{R})||^q \right)^{1/q} \\
&\sim m^{1/q}.
\end{aligned}$$

The equivalence in (30) turns out to be an immediate consequence of Proposition 3 and part (i). To verify (29) observe

$$||f_m|S^s_{p,q}F(\mathbb{R}^2)|| \sim 2^{m(1/p-1)} \left\| \left(\sum_{j=0}^{m} |\mathscr{F}^{-1}[p_{j,m-j}]|^q \right)^{1/q} \right| L_p(\mathbb{R}) \right\| \sim m^{1/p}$$

because of (ii). The proof is complete. □

Remark 11. The main ideas of the proof of the preceding lemma are taken from [26, Lemma 1.1 in Chapter 3]. There two-sided estimates of the L_p-norm of the Dirichlet kernel with respect to the hyperbolic cross are given.

Remark 12. Also in this situation one could use the technique of Burenkov and Gol'dman [7] to carry over the estimates from the periodic situation to the nonperiodic one. That would result in the estimate from above in (25). For that reason we decided to give a complete proof by repeating (partially) the arguments from the periodic case. Vice versa, referring again to [7], the contents of Lemma 3 can be carried over to its periodic analog.

Even more simple are the following nonperiodic counterparts of lacunary series. We start with a function $\psi \in \mathscr{S}(\mathbb{R}^2)$ satisfying $\operatorname{supp} \mathscr{F}\psi \subset \{\xi : |\xi| \leqslant 1\}$.

Then we put

$$f_\alpha(x) = \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \alpha_{j,k} \mathscr{F}^{-1}[\psi(\xi_1 - c2^j, \xi_2 - c2^k)](x) \tag{32}$$

for a given sequence $\alpha = \{\alpha_{j,k}\}_{j,k}$ of complex numbers. For $c = 7/8$ (cf. (5)) we have

$$\mathscr{F}^{-1}[\varphi_j(\xi_1)\varphi_k(\xi_2)\mathscr{F} f_\alpha(\xi_1, \xi_2)](x) = \alpha_{j,k}(\mathscr{F}^{-1}\psi)(x) e^{i(c2^j x_1 + c2^k x_2)}$$

if $j, k \geqslant 3$. This implies the following.

Lemma 4. *Let* $1 \leqslant q \leqslant \infty$.

(i) *Suppose* $1 \leqslant p \leqslant \infty$. *We have*

$$\begin{aligned} ||f_\alpha | S^r_{p,q} F(\mathbb{R}^2)|| &= ||f_\alpha | S^r_{p,q} B(\mathbb{R}^2)|| \\ &= ||\mathscr{F}^{-1}\psi | L_p(\mathbb{R}^2)|| \left(\sum_{j=3}^{\infty} \sum_{k=3}^{\infty} 2^{(j+k)sq} |\alpha_{j,k}|^q \right)^{1/q}. \end{aligned}$$

(ii) *Suppose* $1 < p < \infty$. *Then there exist positive constants* c_1 *and* c_2 *such that*

$$c_1 |||f_\alpha | A^r_{p,q}(\mathbb{R}^2)||| \leqslant \left(\sum_{m=3}^{\infty} 2^{smq} \left[\sum_{j=3}^{m-2} |\alpha_{j,m-j+1}|^2 \right]^{q/2} \right)^{1/q} \leqslant c_2 |||f_\alpha | A^r_{p,q}(\mathbb{R}^2)|||.$$

Proof. Part (i) is obvious. To show (ii) we use Proposition 3. We have

$$||S^H_{m+1} f_\alpha - S^H_m f_\alpha | L_p(\mathbb{R}^2)|| \sim \left(\sum_{j+k=m+1} |\alpha_{j,k}|^2 \right)^{1/2}$$

by the Littlewood–Paley assertion (16). From that the claim follows. □

5.2. Horizontal embeddings

We fix the smoothness parameter r and the integrability parameter p and investigate the influence of the microscopic parameter q.

Theorem 4. *Suppose* $1 < p < \infty$, $1 \leqslant q,\ u \leqslant \infty$, *and* $r > 0$.

(i) *Let* $q < p$. *Then* $S^r_{p,1} F(\mathbb{R}^2) \not\subset A^r_{p,q}$ *holds.*
(ii) *Let* $q \geqslant p$. $S^r_{p,u} F(\mathbb{R}^2) \hookrightarrow A^r_{p,q}$ *holds if and only if* $u \leqslant \min(2, q)$.
(iii) *Let* $q > p$. *Then* $A^r_{p,q} \not\subset S^r_{p,\infty} F(\mathbb{R}^2)$ *holds.*
(iv) *The embedding* $A^r_{p,q} \hookrightarrow S^r_{p,u} F$ *holds if and only if* $q \leqslant p$ *and* $u \geqslant \max(2, q)$.

Proof.

Step 1: Proof of (i). Remarks 3 and 8 show that the embedding $S^r_{p,1}F(\mathbb{R}^2) \hookrightarrow A^r_{p,q}(\mathbb{R}^2)$ would imply $F^r_{p,1}(\mathbb{R}) \hookrightarrow B^r_{p,q}(\mathbb{R})$. But this is known to be not true under these restrictions, cf. [23].

Step 2: Proof of (iii). We argue as in Step 1. The embedding $A^r_{p,q}(\mathbb{R}^2) \hookrightarrow S^r_{p,\infty}F(\mathbb{R}^2)$ would imply $B^r_{p,q}(\mathbb{R}) \hookrightarrow F^r_{p,\infty}(\mathbb{R})$. But this is known to be not true under these restrictions, cf. [23].

Step 3: Proof of (ii). *Sufficiency*. We employ the elementary inequality (12) and the Littlewood–Paley Theorem. Using $q \geqslant p$ this leads to

$$\left\| 2^{rm} \sum_{j=0}^{m} \widetilde{f}_{j,m-j} \Big| \ell_q(L_p(\mathbb{R}^2)) \right\| \leqslant (1/A_p) \left\| \left(\sum_{m=1}^{\infty} 2^{rmq} \left(\sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^2 \right)^{q/2} \right)^{1/q} \Big| L_p(\mathbb{R}^2) \right\| \leqslant (1/A_p) \left\| \left(\sum_{m=1}^{\infty} \sum_{j=0}^{m} 2^{rmu} |\widetilde{f}_{j,m-j}|^u \right)^{1/u} \Big| L_p(\mathbb{R}^2) \right\|$$

which guarantees the continuous embedding $S^r_{p,u}F(\mathbb{R}^2) \hookrightarrow A^r_{p,q}(\mathbb{R}^2)$ in view of Proposition 3.

Step 4: Proof of (ii). *Necessity*. We employ the family of functions from (32). With $\beta_{j,k} = 2^{(j+k)r}\alpha_{j,k}$ the embedding $S^r_{p,u}F(\mathbb{R}^2) \hookrightarrow A^r_{p,q}$ yields the validity of

$$\left(\sum_{m=1}^{\infty} \left(\sum_{j+k=m+1} |\beta_{j,k}|^2 \right)^{q/2} \right)^{1/q} \leqslant C \left(\sum_{j=3}^{\infty} \sum_{k=3}^{\infty} |\beta_{j,k}|^u \right)^{1/u}$$

for arbitrary $\beta_{j,k}$, cf. Lemma 4. But this is true if and only if $u \leqslant \min(2,q)$.

Step 5: Proof of (iv). *Sufficiency*. Suppose $q \leqslant 2 \leqslant p$ and let $u = 2$. We have

$$\left\| \left(\sum_{m=1}^{\infty} 2^{2rm} \sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^2 \right)^{1/2} \Big| L_p(\mathbb{R}^2) \right\| \leqslant \left(\sum_{m=1}^{\infty} 2^{2rm} \left\| \left(\sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^2 \right)^{1/2} \Big| L_p(\mathbb{R}^2) \right\|^2 \right)^{1/2} \leqslant B_p \left(\sum_{m=1}^{\infty} 2^{2rm} \left\| \sum_{j=0}^{m} \widetilde{f}_{j,m-j} \Big| L_p(\mathbb{R}^2) \right\|^2 \right)^{1/2},$$

cf. (12) and (16). The claim follows from the monotonicity of the ℓ_q-norms, Proposition 3 and the Littlewood–Paley assertion (16). Now we consider the case

$2 \leqslant q \leqslant p$ and $u = q$. Then

$$\left\| \left(\sum_{m=1}^{\infty} 2^{rmq} \sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^q \right)^{1/q} \middle| L_p(\mathbb{R}^2) \right\|$$

$$\leqslant \left(\sum_{m=1}^{\infty} 2^{rmq} \left\| \left(\sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^q \right)^{1/q} \middle| L_p(\mathbb{R}^2) \right\|^q \right)^{1/q}$$

$$\leqslant \left(\sum_{m=1}^{\infty} 2^{rmq} \left\| \left(\sum_{j=0}^{m} |\widetilde{f}_{j,m-j}|^2 \right)^{1/2} \middle| L_p(\mathbb{R}^2) \right\|^q \right)^{1/q}$$

$$\leqslant B_p \left(\sum_{m=1}^{\infty} 2^{rmq} \left\| \sum_{j=0}^{m} \widetilde{f}_{j,m-j} \middle| L_p(\mathbb{R}^2) \right\|^q \right)^{1/q}.$$

Step 6: Proof of (iv). *Necessity*. An application of Lemma 4 shows that the embedding $A^r_{p,q}(\mathbb{R}^2) \hookrightarrow S^r_{p,u}F(\mathbb{R}^2)$ implies the existence of a general constant C such that

$$\|\beta_{j,k}|\ell_u\| \leqslant C \left(\sum_{m=1}^{\infty} \left[\sum_{j=3}^{m-2} |\beta_{j,m-j+1}|^2 \right]^{q/2} \right)^{1/q}$$

holds for all sequences $\{\beta_{j,k}\}_{j,k}$. But this implies $u \geqslant \max(q, 2)$. The necessity of $q \leqslant p$ follows from part (iii). □

Theorem 5. *Suppose* $1 < p < \infty$, $1 \leqslant q,\ u \leqslant \infty$, *and* $r > 0$.

(i) *The embedding* $S^r_{p,u}B(\mathbb{R}^2) \hookrightarrow A^r_{p,q}(\mathbb{R}^2)$ *holds if and only if* $u \leqslant \min(2, q, p)$.
(ii) *The embedding* $A^r_{p,q}(\mathbb{R}^2) \hookrightarrow S^r_{p,u}B(\mathbb{R}^2)$ *holds if and only if* $u \geqslant \max(2, q, p)$.

Proof.

Step 1: Proof of (i). *Necessity*. The necessity of $u \leqslant \min(2, q)$ follows as in Step 6 of the proof of Theorem 4. To prove the necessity of $u \leqslant p$ we consider the functions introduced in (27). From (28) and (30) the desired result follows.

Step 2: Proof of (ii). *Necessity*. The necessity of $u \geqslant p$ follows from (28) and (30). Further, the necessity of $u \geqslant \max(2, q)$ can be derived by using Lemma 4.

Step 3: Proof of (i). *Sufficiency*. We make use of our elementary inequality (12) and of the Littlewood–Paley assertion. With $u \leqslant t = \min(2, p, q)$ we find

$$\left(\sum_{m=0}^{\infty} 2^{mrq} \|S^H_{m+1}f - S^H_m f|L_p(\mathbb{R}^2)\|^q \right)^{1/q}$$

$$\leqslant c\left(\sum_{m=0}^{\infty} 2^{mrq}\left\|\left(\sum_{j+k=m+1} |\widetilde{f}_{j,k}|^2\right)^{1/2}\Big|L_p(\mathbb{R}^2)\right\|^q\right)^{1/q}$$

$$\leqslant \left(\sum_{m=0}^{\infty} 2^{mrq}\left(\sum_{j+k=m+1} \|\widetilde{f}_{j,k}|L_p(\mathbb{R}^2)\|^t\right)^{q/t}\right)^{1/q}$$

$$\leqslant \|f|S^r_{p,u}B(\mathbb{R}^2)\|.$$

Step 4: Proof of (ii). *Sufficiency*. Using $\max(2,p)\leqslant u$ we find $\|\cdot|\ell_u(L_p(\mathbb{R}^2))\|\leqslant\|\cdot|L_p(\mathbb{R}^2)(\ell_2)\|$ (see (12)). Now we apply (16)

$$\left(\sum_{m=0}^{\infty} 2^{rmu}\sum_{j=0}^{m}\|\widetilde{f}_{j,m-j}|L_p(\mathbb{R}^2)\|^u\right)^{1/u} \leqslant \left(\sum_{m=0}^{\infty} 2^{rmu}\left\|\left[\sum_{j=0}^{m}|\widetilde{f}_{j,m-j}|^2\right]^{1/2}\Big|L_p(\mathbb{R}^2)\right\|^u\right)^{1/u}$$

$$\leqslant B_p\left(\sum_{m=0}^{\infty} 2^{rmu}\left\|\sum_{j=0}^{m}\widetilde{f}_{j,m-j}|L_p(\mathbb{R}^2)\right\|^u\right)^{1/u}.$$

The final step consists in using the monotonicity of the ℓ_u-spaces and $u\geqslant q$. □

Of particular interest is the case $q=\infty$. We formulate some simple consequences of Theorems 4 and 5 and 3(i).

Corollary 1. *Suppose* $1<p<\infty$, $1\leqslant q$, $u\leqslant\infty$, *and* $r>0$.

(i) *The embedding* $S^r_{p,u}B(\mathbb{R}^2)\hookrightarrow A^r_{p,\infty}(\mathbb{R}^2)$ *holds if and only if* $u\leqslant\min(2,p)$.
(ii) *The embedding* $S^r_{p,u}F(\mathbb{R}^2)\hookrightarrow A^r_{p,\infty}(\mathbb{R}^2)$ *holds if and only if* $u\leqslant 2$.
(iii) *Whenever* $S^r_{p,u}B(\mathbb{R}^2)\hookrightarrow A^r_{p,\infty}(\mathbb{R}^2)$ *holds, then* $S^r_{p,u}B(\mathbb{R}^2)\hookrightarrow S^r_{p,2}F(\mathbb{R}^2)$.

Remark 13. Hence, within the scales of Besov and Lizorkin–Triebel classes the optimal embeddings for $A^r_{p,\infty}(\mathbb{R}^2)$ are:

$$S^r_{p,2}F(\mathbb{R}^2)\hookrightarrow A^r_{p,\infty}(\mathbb{R}^2)\hookrightarrow S^r_{p,\infty}B(\mathbb{R}^2). \tag{33}$$

The right-hand side in this formula can be found in Lizorkin and Nikol'skij [17, Theorem 4.1]. The "if-parts" of (33) have been known in different contexts, cf. [9,15,25,26].

Quite similarly one can deal with the other extremal case $A^r_{p,1}(\mathbb{R}^2)$.

Corollary 2. *Suppose* $1<p<\infty$, $1\leqslant q$, $u\leqslant\infty$, *and* $r>0$.

(i) *The embedding* $A^r_{p,1}(\mathbb{R}^2)\hookrightarrow S^r_{p,u}B(\mathbb{R}^2)$ *holds if and only if* $u\geqslant\max(2,p)$.
(ii) *The embedding* $A^r_{p,1}(\mathbb{R}^2)\hookrightarrow S^r_{p,u}F(\mathbb{R}^2)$ *holds if and only if* $u\geqslant 2$.
(iii) *Whenever* $A^r_{p,1}(\mathbb{R}^2)\hookrightarrow S^r_{p,u}B(\mathbb{R}^2)$ *holds, then* $A^r_{p,1}(\mathbb{R}^2)\hookrightarrow S^r_{p,2}F(\mathbb{R}^2)\hookrightarrow S^r_{p,u}B(\mathbb{R}^2)$.

Remark 14. The only Besov space which is a subspace of $A^r_{p,1}(\mathbb{R}^2)$ is given by $S^r_{p,1}B(\mathbb{R}^2)$, cf. Theorem 5. Hence, within the scales of Besov and Lizorkin–Triebel classes the optimal embeddings for $A^r_{p,1}(\mathbb{R}^2)$ are:

$$S^r_{p,1}B(\mathbb{R}^2) \hookrightarrow A^r_{p,1}(\mathbb{R}^2) \hookrightarrow S^r_{p,2}F(\mathbb{R}^2).$$

Remark 15. Lizorkin and Nikol'skij [17] investigated the relations between $A^r_{p,q}(\mathbb{R}^2)$ and $S^r_{p,q}B(\mathbb{R}^2)$ (with coincidence of the microscopic parameter q).

Based on the above investigations it is now quite easy to understand under which conditions the approximation classes are particular spaces of Besov or Lizorkin–Triebel type of dominating mixed smoothness.

Corollary 3. *Suppose* $1 < p_0, p_1 < \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, $r_0 > 0$ *and* $r_1, r_2 \in \mathbb{R}$.

(i) *The classes* $A^{r_0}_{p_0,q_0}(\mathbb{R}^2)$ *and* $S^{r_1}_{p_1,q_1}B(\mathbb{R}^2)$ *coincide if and only if* $r_0 = r_1$, $p_0 = p_1 = 2$, *and* $q_0 = q_1 = 2$.
(ii) *The classes* $A^{r_0}_{p_0,q_0}(\mathbb{R}^2)$ *and* $S^{r_1}_{p_1,q_1}F(\mathbb{R}^2)$ *coincide if and only if* $r_0 = r_1$, $p_0 = p_1 = 2$, *and* $q_0 = q_1 = 2$.

Proof.
Step 1: Let $p = q = 2$. Then we may apply Proposition 3 and use the pairwise orthogonality of the $\widetilde{f}_{j,k}$ to obtain

$$||S^H_{m+1}f - S^H_m f|L_2(\mathbb{R}^2)||^2 = \sum_{j+k=m+1} ||\widetilde{f}_{j,k}|L_2(\mathbb{R}^2)||^2.$$

Step 2: We assume coincidence. Then, using functions of type (32) we derive $r_0 = r_1$. Considering functions of type (27) we conclude $p_0 = p_1$ whenever both belong to $(1, \infty)$. It remains to clarify the q-dependence. But here we can use Theorems 4 and 5 to prove the claim. □

5.3. Diagonal embeddings

As in case of Besov and Lizorkin–Triebel classes we shall derive diagonal embeddings for the approximation spaces and also mixed assertions of such a type. Our main tool will be real interpolation.

Diagonal embeddings in the isotropic situation are well-understood, cf. e.g. [23]. Taking into account Remarks 3 and 8 this implies that the embedding

$$X^{r_0}_{p_0,q_0} \hookrightarrow Y^{r_1}_{p_1,q_1}$$

(here X, Y denote elements of either the $S-B$-, $S-F$- or A-scale) implies

$$p_0 \leqslant p_1 \quad \text{and} \quad r_0 - \frac{1}{p_0} \geqslant r_1 - \frac{1}{p_1}.$$

For $p_0 < p_1$ we only deal with the limiting situation $r_0 - 1/p_0 = r_1 - 1/p_1$.

Theorem 6. (i) *Let* $r_0 > 0$, $r \in \mathbb{R}$, *and* $1 < p_0 < p < \infty$, *such that*

$$r_0 - \frac{1}{p_0} = r - \frac{1}{p}.$$

Then it holds

$$A^{r_0}_{p_0,p}(\mathbb{R}^2) \hookrightarrow S^r_{p,q}F(\mathbb{R}^2) \tag{34}$$

for all q, $1 \leqslant q \leqslant \infty$. *Furthermore* $A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \not\subset S^r_{p,\infty}F(\mathbb{R}^2)$ *if* $q_0 > p$.

(ii) *Let* $r, r_1 > 0$ *and* $1 < p < p_1 < \infty$, *such that*

$$r - \frac{1}{p} = r_1 - \frac{1}{p_1}.$$

Then it holds

$$S^r_{p,q}F(\mathbb{R}^2) \hookrightarrow A^{r_1}_{p_1,p}(\mathbb{R}^2) \tag{35}$$

for all q, $1 \leqslant q \leqslant \infty$. *Furthermore* $S^r_{p,1}F(\mathbb{R}^2) \not\subset A^{r_1}_{p_1,q_1}(\mathbb{R}^2)$ *if* $q_1 < p$.

Proof.

Step 1: To prove (34) we may assume $q = 1$. Let r_0, r and $1 < p_0 < p$ be given. We choose p^* and p^{**} satisfying $p_0 < p^{**} < p < p^* < \infty$. Next, we choose Θ, r^*, and r^{**} such that

$$\frac{1}{p} = \frac{1-\Theta}{p^*} + \frac{\Theta}{p^{**}},$$
$$r^* - \frac{1}{p_0} = r - \frac{1}{p^*},$$
$$r^{**} - \frac{1}{p_0} = r - \frac{1}{p^{**}}.$$

Then we have $r_0 = (1-\Theta)r^* + \Theta r^{**}$, see Fig. 1.

Theorem 2(ii) yields

$$S^{r^*}_{p_0,2}F(\mathbb{R}^2) \hookrightarrow S^r_{p^*,1}F(\mathbb{R}^2) \quad \text{and} \quad S^{r^{**}}_{p_0,2}F(\mathbb{R}^2) \hookrightarrow S^r_{p^{**},1}F(\mathbb{R}^2).$$

By real interpolation, cf. Proposition 5, it follows therefrom

$$\begin{aligned} A^{r_0}_{p_0,p}(\mathbb{R}^2) &= (S^{r^*}_{p_0,2}F(\mathbb{R}^2), S^{r^{**}}_{p_0,2}F(\mathbb{R}^2))_{\Theta,p} \\ &\hookrightarrow (S^r_{p^*,1}F(\mathbb{R}^2), S^r_{p^{**},1}F(\mathbb{R}^2))_{\Theta,p} \\ &\overset{(24)}{\hookrightarrow} S^r_{p,1}F(\mathbb{R}^2). \end{aligned}$$

This proves (34).

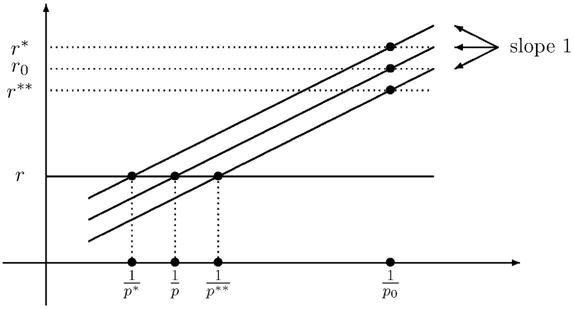

Fig. 1. Real interpolation in Step 1 of the proof.

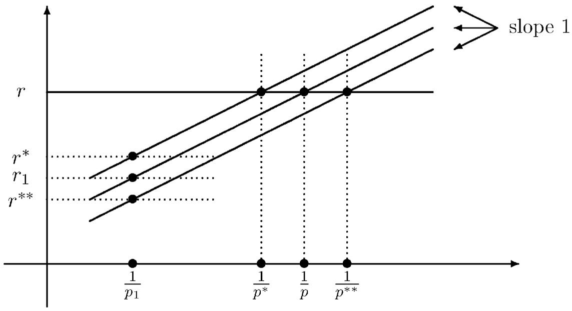

Fig. 2. Real interpolation in Step 2 of the proof.

Step 2: To prove (35) we may assume $q = \infty$. Let r, r_1 and p, p_1 be given. We choose p^* and p^{**} satisfying $1 < p^{**} < p < p^* < p_1 \leqslant \infty$. Next, we choose Θ, r^*, and r^{**} such that

$$\frac{1}{p} = \frac{1-\Theta}{p^*} + \frac{\Theta}{p^{**}},$$
$$r - \frac{1}{p^*} = r^* - \frac{1}{p_1},$$
$$r - \frac{1}{p^{**}} = r^{**} - \frac{1}{p_1}.$$

Then we have $r_1 = (1-\Theta)r^* + \Theta r^{**}$, see Fig. 2.

By real interpolation, cf. Proposition 8, and using Theorem 2 it follows therefrom

$$\begin{aligned} S^r_{p,\infty}F(\mathbb{R}^2) &= (S^r_{p^*,\infty}F(\mathbb{R}^2), S^r_{p^{**},\infty}F(\mathbb{R}^2))_{\Theta,p} \\ &\hookrightarrow (S^{r^*}_{p_1,2}F(\mathbb{R}^2), S^{r^{**}}_{p_1,2}F(\mathbb{R}^2))_{\Theta,p} \\ &= A^{r_1}_{p_1,p}(\mathbb{R}^2). \end{aligned}$$

This proves (35).

Step 3: The optimality of embeddings (34) and (35) follows from the optimality of $B^{r_0}_{p_0,p}(\mathbb{R}) \hookrightarrow F^r_{p,q}(\mathbb{R})$ and of $F^r_{p,q}(\mathbb{R}) \hookrightarrow B^{r_1}_{p_1,p}(\mathbb{R})$, respectively, cf. Remarks 3, 8 and [23]. □

Next, we compare the approximation spaces along these diagonals.

Theorem 7. *Let* $r_0 > 0$, $r \in \mathbb{R}$, *and* $1 < p_0 < p < \infty$, *such that*

$$r_0 - \frac{1}{p_0} = r - \frac{1}{p}.$$

Then it holds

$$A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \hookrightarrow A^r_{p,q}(\mathbb{R}^2) \tag{36}$$

if and only if $q_0 \leqslant q$.

Proof.
Step 1: *Sufficiency*. We use Theorems 6 and 4(ii) to derive

$$A^{r_i}_{p_0,p}(\mathbb{R}^2) \hookrightarrow S^{t_i}_{p,1}F(\mathbb{R}^2) \hookrightarrow A^{t_i}_{p,p}(\mathbb{R}^2), \quad p_0 < p, \quad r_i - \frac{1}{p_0} = t_i - \frac{1}{p}, \quad i = 1,2.$$

The general result follows now by real interpolation because of

$$A^{r_0}_{p_0,q}(\mathbb{R}^2) = (A^{r_1}_{p_0,p}(\mathbb{R}^2), A^{r_2}_{p_0,p}(\mathbb{R}^2))_{\Theta,q} \hookrightarrow (A^{t_1}_{p,p}(\mathbb{R}^2), A^{t_2}_{p,p}(\mathbb{R}^2))_{\Theta,q} = A^r_{p,q}(\mathbb{R}^2),$$

where $r_0 = (1-\Theta)r_1 + \Theta r_2$, cf. Proposition 6.
Step 2: *Necessity*. We use Remark 8 and the necessity of $q_0 \leqslant q$ in case of the embedding $B^{r_0}_{p_0,q_0}(\mathbb{R}) \hookrightarrow B^r_{p,q}(\mathbb{R})$, cf. [23]. □

Remark 16. Theorem 6 shows that in Theorem 3 the spaces $S^{r_i}_{p_i,q_i}B(\mathbb{R}^2)$ can be replaced by $A^{r_i}_{p_i,q_i}(\mathbb{R}^2)$, $i = 0,1$. Moreover, it follows from Theorem 7 that the approximation spaces themselves behave like the spaces $S^r_{p,q}B(\mathbb{R}^2)$ with respect to embeddings with constant differential dimension (cf. Theorem 2).

A comparison of the approximation spaces with the Besov spaces is more sophisticated.

Theorem 8. *Let* $r_0 > 0$, $r \in \mathbb{R}$, $1 \leqslant q,\ q_0 \leqslant \infty$, *and* $1 < p_0 < p < \infty$, *such that*

$$r_0 - \frac{1}{p_0} = r - \frac{1}{p}.$$

Then it holds

$$A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \hookrightarrow S^r_{p,q}B(\mathbb{R}^2) \tag{37}$$

if and only if $\max(p_0, q_0) \leqslant q$.

Proof. To begin with we investigate sufficiency.

Step 1: Let $q_0 \leqslant p_0$. Then Theorems 4(iv) and 3(ii) yield

$$A^{r_0}_{p_0,p_0}(\mathbb{R}^2) \hookrightarrow S^{r_0}_{p_0,\infty}F(\mathbb{R}^2) \hookrightarrow S^r_{p,p_0}B(\mathbb{R}^2).$$

Hence, if $q \geqslant p_0$ we have $A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \hookrightarrow S^r_{p,q}B(\mathbb{R}^2)$.

Step 2: Let $p_0 < q_0$ and $q_0 \geqslant 2$. Then Theorems 5 and 2(i) yield

$$A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \hookrightarrow S^{r_0}_{p_0,\max(2,q_0)}B(\mathbb{R}^2) \hookrightarrow S^r_{p,q_0}B(\mathbb{R}^2).$$

Hence, if $q \geqslant q_0$ we have $A^{r_0}_{p_0,q_0}(\mathbb{R}^2) \hookrightarrow S^r_{p,q}B(\mathbb{R}^2)$.

Step 3: It remains to consider $1 < p_0 < q_0 < 2$. Now we use real interpolation. We choose $0 < \Theta < 1$ such that $1/q_0 = (1-\Theta)/2 + \Theta/p_0$. Next, we choose positive numbers r_1, r_2, t_1 and t_2 such that

$$r_0 = (1-\Theta)r_1 + \Theta r_2, \quad r_1 \neq r_2, \quad r = (1-\Theta)t_1 + \Theta t_2, \quad t_1 \neq t_2$$

and

$$r_1 - \frac{1}{p_0} = t_1 - \frac{1}{p}, \quad r_2 - \frac{1}{p_0} = t_2 - \frac{1}{p}.$$

Then

$$(A^{r_1}_{p_0,1}(\mathbb{R}^2), A^{r_2}_{p_0,1}(\mathbb{R}^2))_{\Theta,q_0} = A^{r_0}_{p_0,q_0}(\mathbb{R}^2),$$

cf. Proposition 6. Furthermore, Proposition 7 yields

$$(S^{t_1}_{p,2}B(\mathbb{R}^2), S^{t_2}_{p,p_0}B(\mathbb{R}^2))_{\Theta,q_0} = S^r_{p,q_0}B(\mathbb{R}^2).$$

Observe

$$A^{r_1}_{p_0,1}(\mathbb{R}^2) \hookrightarrow S^{t_1}_{p,p_0}B(\mathbb{R}^2) \hookrightarrow S^{t_1}_{p,2}B(\mathbb{R}^2) \quad \text{and} \quad A^{r_2}_{p_0,1}(\mathbb{R}^2) \hookrightarrow S^{t_2}_{p,p_0}B(\mathbb{R}^2),$$

see Step 1. The monotonicity of the real method yields the conclusion.

Step 4: *Necessity*. Our standard argument, cf. Remarks 3 and 8, yields the necessity of $q_0 \leqslant q$. Further, our test functions from Lemma 3 prove the necessity of $p_0 \leqslant q$. □

Theorem 9. *Let $r, r_1 > 0$ and $1 < p < p_1 < \infty$, such that*

$$r - \frac{1}{p} = r_1 - \frac{1}{p_1}.$$

(i) *If $1 \leqslant q \leqslant p$ it holds*

$$S^r_{p,q}B(\mathbb{R}^2) \hookrightarrow A^{r_1}_{p_1,q}(\mathbb{R}^2). \tag{38}$$

If $q_1 < q$, then $S^r_{p,q}B(\mathbb{R}^2) \not\subset A^{r_1}_{p_1,q_1}(\mathbb{R}^2)$.

(ii) *Suppose $p < q < \infty$. We have*

$$S^r_{p,q}B(\mathbb{R}^2) \hookrightarrow A^{r_1}_{q,q}(\mathbb{R}^2). \tag{39}$$

If $p < p_1 < q$, then $S^r_{p,q}B(\mathbb{R}^2) \not\subset A^{r_1}_{p_1,\infty}(\mathbb{R}^2)$.

Proof.

Step 1: Proof of (i). We already know

$$S^r_{p,1}B(\mathbb{R}^2) \hookrightarrow A^r_{p,1}(\mathbb{R}^2) \hookrightarrow A^{r_1}_{p_1,1}(\mathbb{R}^2) \quad \text{and}$$
$$S^r_{p,p}B(\mathbb{R}^2) = S^r_{p,p}F(\mathbb{R}^2) \hookrightarrow A^{r_1}_{p_1,p}(\mathbb{R}^2), \tag{40}$$

where we used Theorems 5(i), 7, and 6(ii). Now we continue with real interpolation. For given q, $1<q<p$, we choose $0<\Theta<1$ such that $1/q = (1-\Theta) + \Theta/p$. Next, we choose positive numbers r_2, r_3, t_1 and t_2 such that

$$r = (1-\Theta)t_1 + \Theta t_2, \quad t_1 \neq t_2, \qquad r_1 = (1-\Theta)r_2 + \Theta r_3, \quad r_2 \neq r_3$$

and

$$t_1 - \frac{1}{p} = r_2 - \frac{1}{p_1}, \quad t_2 - \frac{1}{p} = r_3 - \frac{1}{p_1}.$$

Then

$$(S^{t_1}_{p,1}B(\mathbb{R}^2), S^{t_2}_{p,p}B(\mathbb{R}^2))_{\Theta,q} = S^r_{p,q}B(\mathbb{R}^2)$$

and

$$(A^{r_2}_{p_1,1}(\mathbb{R}^2), A^{r_3}_{p_1,p}(\mathbb{R}^2))_{\Theta,q} = A^{r_1}_{p_1,q}(\mathbb{R}^2),$$

cf. Propositions 7 and 6, respectively. The monotonicity of the real method and (40) yield the conclusion.

The second statement in part (i) follows from Remarks 3 and 8.

Step 2: Proof of (ii). Sufficiency follows from

$$S^r_{p,q}B(\mathbb{R}^2) \hookrightarrow S^{r_1}_{q,1}F(\mathbb{R}^2) \hookrightarrow A^{r_1}_{q,q}(\mathbb{R}^2), \tag{41}$$

cf. Theorems 3(i) and 4(ii). From the embedding $S^r_{p,q}B(\mathbb{R}^2) \hookrightarrow A^{r_1}_{p_1,\infty}(\mathbb{R}^2)$ and Lemma 3 we derive $q \leqslant p_1$. $\square$

5.4. Applications to approximation in different metrics

Let X_p be either $A^r_{p,q}(\mathbb{R}^2)$ or $S^r_{p,q}F(\mathbb{R}^2)$ or $S^r_{p,q}B(\mathbb{R}^2)$. We are dealing with the question which of these classes have the property that $f \in L_{p_1}(\mathbb{R}^2)$ and

$$E_m(f, L_{p_1}) \leqslant c_f 2^{-mr_1}, \quad m = 1, 2, \ldots, \tag{42}$$

where $r_1 > 0$ is given. Of course, this holds if and only if $X_p \hookrightarrow A^{r_1}_{p_1,\infty}(\mathbb{R}^2)$. In view of the preceding subsection the answer is easy, now.

Corollary 4. *Let* $1<p<p_1<\infty$ *and* $r_1>0$. *Let* $1 \leqslant q \leqslant \infty$.

(i) (42) *holds for all* $f \in S^r_{p,q}F(\mathbb{R}^2)$ *if and only if*

$$r \geqslant r_1 - \left(\frac{1}{p_1} - \frac{1}{p}\right). \tag{43}$$

(ii) *Eq.* (42) *holds for all* $f \in S^r_{p,q}B(\mathbb{R}^2)$ *if and only if either*

$$r > r_1 - \left(\frac{1}{p_1} - \frac{1}{p}\right)$$

and q is arbitrary or

$$r = r_1 - \left(\frac{1}{p_1} - \frac{1}{p}\right).$$

and $q \leqslant p_1$. *Moreover, if* (42) *holds for all* $f \in S^r_{p,q}B(\mathbb{R}^2)$, *then*

$$\lim_{m \to \infty} 2^{mr_1} E_m(f, L_{p_1}) = 0$$

for all $f \in S^r_{p,q}B(\mathbb{R}^2)$.

(iii) (42) *holds for all* $f \in A^r_{p,q}(\mathbb{R}^2)$ *if and only if* (43) *holds.*

For better reference we state also the embeddings of $A^r_{p,q}(\mathbb{R}^2)$ into $L_{p_1}(\mathbb{R}^2)$ and $C(\mathbb{R}^2)$, respectively. Here $C(\mathbb{R}^2)$ denotes the space of uniformly continuous and bounded functions on $\mathbb{R}^2$ equipped with the supremum norm.

Corollary 5. *Let* $1 < p < p_1 < \infty$ *and* $1 \leqslant q \leqslant \infty$.

(i) *The embedding* $A^r_{p,q}(\mathbb{R}^2) \hookrightarrow L_{p_1}(\mathbb{R}^2)$ *holds if and only if either*

$$r > \frac{1}{p} - \frac{1}{p_1}$$

and q is arbitrary or

$$r = \frac{1}{p} - \frac{1}{p_1} \quad \text{and} \quad q \leqslant p_1.$$

(ii) *The embedding* $A^r_{p,q}(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$ *holds if and only if* $r > 1/p$.

Proof. Part (i) is a consequence of $S^0_{p_1,2}F(\mathbb{R}^2) = L_{p_1}(\mathbb{R}^2)$ (equivalent norms), cf. Remark 2 and Theorem 6. The sufficiency part in statement (ii) follows from

$$A^r_{p,\infty}(\mathbb{R}^2) \hookrightarrow S^r_{p,\infty}B(\mathbb{R}^2) \hookrightarrow S^{1/p}_{p,1}B(\mathbb{R}^2) \hookrightarrow S^0_{\infty,1}B(\mathbb{R}^2), \quad r > 1/p,$$

cf. Theorems 5(ii) and 2(i). To derive the necessity we shall use once again our test functions defined in (27): we have

$$f_m(x) = \frac{2}{\pi} \sum_{j=0}^{m} 2^{-rm} 2^{m(1/p-1)} e^{i3\pi(2^{j-1}x_1 + 2^{m-j-1}x_2)} \frac{\sin(2^{j-1}\pi x_1)\sin(2^{m-j-1}\pi x_2)}{x_1 x_2}, \quad m \in \mathbb{N}.$$

Obviously, if $r = 1/p$, then

$$\|f_m|L_\infty(\mathbb{R}^2)\| = |f_m(0,0)| = 2\pi 2^{-m} \sum_{j=0}^{m} 2^{j-1+m-j-1} = \frac{\pi}{2}(m+1).$$

Lemma 3 shows that $A^{1/p}_{p,1}(\mathbb{R}^2) \not\subset C(\mathbb{R}^2)$ if $1<p<\infty$. □

Remark 17. In view of part (ii) of Corollary 5 it becomes clear that $A^{1/p}_{p,1}(\mathbb{R}^2)$ contains unbounded functions and hence, is quite different from $S^{1/p}_{p,1}B(\mathbb{R}^2)$.

6. Approximation by partial sums with respect to hyperbolic crosses

The aim of this subsection consists in a detailed investigation of the norm of the operators $I - S^H_m$, considered as a mapping from $S^r_{p,q}F(\mathbb{R}^2)$ into $L_p(\mathbb{R}^2)$ and from $S^r_{p,q}B(\mathbb{R}^2)$ into $L_p(\mathbb{R}^2)$. Because of Corollary 1 this is of interest only if $q>2$ in the case of the F-spaces and $q>\min(p,2)$ in case of the B-spaces. Otherwise the above spaces are embedded into $A^r_{p,\infty}(\mathbb{R}^2)$ and it follows that

$$\|f - S^H_m f|L_p(\mathbb{R}^2)\| \leqslant c(f) 2^{-mr}.$$

Step 2 in the proof of Theorem 10 and steps 3 and 4 in the proof of Theorem 11 show that even

$$\|I - S^H_m\colon\, S^r_{p,q}F(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)\| \sim 2^{-mr} \quad (\text{if } q>2),$$

$$\|f - S^H_m f\colon\, S^r_{p,q}B(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)\| \sim 2^{-mr} \quad (\text{if } q>\min(p,2)).$$

All these estimates of $\|I - S^H_m\|$ will be used in a forthcoming paper [22] of the second named author to evaluate the quality of a family of sampling operators related to sparse grids.

Theorem 10. *Suppose* $1<p<\infty$, $2<q\leqslant\infty$, *and* $r>0$. *Then*

$$\|I - S^H_m : S^r_{p,q}F(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)\| \sim m^{\frac{1}{2}-\frac{1}{q}} 2^{-rm}. \tag{44}$$

Proof.

Step 1: Let $1/q + 1/v = 1/2$. (16) and Hölder's inequality yield

$$\|f - S^H_m f|L_p(\mathbb{R}^2)\|$$

$$\leqslant (1/A_p) \left\| \left(\sum_{j+k>m} |\widetilde{f}_{j,k}|^2 \right)^{1/2} \middle| L_p(\mathbb{R}^2) \right\|$$

$$\leqslant (1/A_p) 2^{-mr} \left\| \left(\sum_{j+k>m} 2^{(-r(j+k)+mr)2} 2^{r(j+k)2} |\widetilde{f}_{j,k}|^2 \right)^{1/2} \Bigg| L_p(\mathbb{R}^2) \right\|$$

$$\leqslant (1/A_p) 2^{-mr} \left(\sum_{j+k>m} 2^{(-r(j+k)+mr)v} \right)^{1/v} ||f|S^r_{p,q}F(\mathbb{R}^2)||.$$

Now, the assertion on the upper bound follows.

Step 2: To prove the lower bound we test the operator on functions of type (32). Employing the same notations as in the proof of Theorem 4 (Step 6) this yields

$$||2^{r(m+1)}(I - S^H_m) : S^r_{p,q}F(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)|| \geqslant c \frac{\left(\sum_{j+k>m} 2^{r(m+1-(j+k))2} |\beta_{j,k}|^2 \right)^{1/2}}{||\beta_{j,k}|\ell_q||},$$

for some positive c. We choose

$$\beta_{j,k} = \begin{cases} 1, & 0 \leqslant j \leqslant m \quad \text{and} \quad k = m+1-j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $||\beta_{j,k}|\ell_q|| = (m+1)^{1/q}$ and $\left(\sum_{j+k>m} 2^{r(m+1-(j+k))2} |\beta_{j,k}|^2 \right)^{1/2} = (m+1)^{1/2}$. □

Theorem 11. *Let* $r > 0$.

(i) *Suppose* $1 < p \leqslant 2$ *and* $p \leqslant q \leqslant \infty$. *Then*

$$||I - S^H_m : S^r_{p,q}B(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)|| \sim m^{\frac{1}{p}-\frac{1}{q}} 2^{-rm}. \tag{45}$$

(ii) *Suppose* $2 < p < \infty$ *and* $q > 2$. *Then*

$$||I - S^H_m : S^r_{p,q}B(\mathbb{R}^2) \mapsto L_p(\mathbb{R}^2)|| \sim m^{\frac{1}{2}-\frac{1}{q}} 2^{-rm}. \tag{46}$$

Proof.

Step 1: We prove the estimate from above in (45). Observe

$$\left\| \sum_{j=m+1}^{\infty} \sum_{k=0}^{\infty} \widetilde{f}_{j,k} \Bigg| L_p(\mathbb{R}^2) \right\| \leqslant 2^{-mr} \sum_{j=m+1}^{\infty} \sum_{k=0}^{\infty} 2^{-r(j+k)+mr} 2^{r(j+k)} ||\widetilde{f}_{j,k}|L_p(\mathbb{R}^2)||$$
$$\leqslant c 2^{-mr} ||f|S^r_{p,\infty}B(\mathbb{R}^2)||.$$

Analogously

$$\left\| \sum_{k=m+1}^{\infty} \sum_{j=0}^{\infty} \widetilde{f}_{j,k} \Bigg| L_p(\mathbb{R}^2) \right\| \leqslant c 2^{-mr} ||f|S^r_{p,\infty}B(\mathbb{R}^2)||.$$

Hence, the major part of the operator norm comes from the sum $\sum_{j=0}^{m} \sum_{k=m-j+1}^{m}$. Here by means of the Littlewood–Paley assertion (16), of the elementary inequality

(12) and Hölder's inequality with $1/u + 1/q = 1/p$ we find

$$\begin{aligned}
&\left\|\sum_{j=0}^{m} \sum_{k=m-j+1}^{m} \widetilde{f}_{j,k} \Big| L_p(\mathbb{R}^2)\right\| \\
&\leqslant (1/A_p)\left\|\left(\sum_{j=0}^{m} \sum_{\ell=1}^{m} |\widetilde{f}_{j,m-j+\ell}|^2\right)^{1/2} \Big| L_p(\mathbb{R}^2)\right\| \\
&\leqslant (1/A_p) \sum_{\ell=1}^{m}\left(\sum_{j=0}^{m} ||\widetilde{f}_{j,m-j+\ell}|L_p(\mathbb{R}^2)||^p\right)^{1/p} \\
&\leqslant (1/A_p) \sum_{\ell=1}^{m}\left(\sum_{j=0}^{m} 2^{-r(m+\ell)u}\right)^{1/u} \left(\sum_{j=0}^{m} 2^{r(m+\ell)q}||\widetilde{f}_{j,m-j+\ell}|L_p(\mathbb{R}^2)||^q\right)^{1/q} \\
&\leqslant c2^{-mr}m^{1/u}||f|S^r_{p,q}B(\mathbb{R}^2)||.
\end{aligned}$$

Step 2: We prove the estimate from above in (46). As in Step 1 the major part of the operator norm comes from the sum $\sum_{j=0}^{m}\sum_{k=m-j+1}^{m}$. Using (12) and putting $1/2 = 1/q + 1/u$ we can derive, quite similar to Step 1,

$$\begin{aligned}
&\left\|\sum_{j=0}^{m} \sum_{k=m-j+1}^{m} \widetilde{f}_{j,k} \Big| L_p(\mathbb{R}^2)\right\| \\
&\leqslant (1/A_p) \sum_{\ell=1}^{m}\left(\sum_{j=0}^{m} ||\widetilde{f}_{j,m-j+\ell}|L_p(\mathbb{R}^2)||^2\right)^{1/2} \\
&\leqslant (1/A_p) \sum_{\ell=1}^{m}\left(\sum_{j=0}^{m} 2^{-r(m+\ell)u}\right)^{1/u} \left(\sum_{j=0}^{m} 2^{r(m+\ell)q}||\widetilde{f}_{j,m-j+\ell}|L_p(\mathbb{R}^2)||^q\right)^{1/q} \\
&\leqslant c2^{-mr}m^{1/u}||f|S^r_{p,q}B(\mathbb{R}^2)||.
\end{aligned}$$

Step 3: Estimate from below in (46). Since $||f|S^r_{p,q}F(\mathbb{R}^2)|| = ||f|S^r_{p,q}B(\mathbb{R}^2)||$ for functions of type (32) the argument from Step 2 of the proof of Theorem 10 applies.

Step 4: Estimate from below in (45). We shall test the operator $I - S^H_m$ with the functions f_m defined in (27). Then

$$||f_{m+1} - S^H_m f_{m+1}|L_p(\mathbb{R}^2)|| = ||f_{m+1}|L_p(\mathbb{R}^2)|| \sim m^{1/p}2^{-rm}.$$

Recall, $||f_m|S^r_{p,q}B(\mathbb{R}^2)||$ has been calculated in Lemma 3. In view of the independence of the constants on m the result follows. □

Remark 18. For $q = \infty$ the assertion of the Theorem 10 is known, cf. [17]. In the periodic setting it was known even for a longer time, cf. [6] ($p = 2$), [18], [26, Theorem III.3.3]. The analogous problem for spaces defined on the unit cube and spline approximation has been treated by Kamont [15].

Remark 19. Let us mention that in the periodic context Delvos and Schempp [8] gave estimates of the best approximation in the L_2-norm by using Korobov spaces (defined by the behavior of the Fourier coefficients).

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